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**Abstract**

For a general class of hyperbolic-parabolic systems including the compressible Navier-Stokes and compressible MHD equations, we prove existence and stability of noncharacteristic viscous boundary layers for a variety of boundary conditions including classical Navier-Stokes boundary conditions. Our first main result, using the abstract framework established by the authors in the companion work [GMWZ6], is to show that existence and stability of arbitrary amplitude exact boundary-layer solutions follow from a uniform spectral stability condition on layer profiles that is expressible in terms of an Evans function (uniform Evans stability). Our second is to show that uniform Evans stability for small-amplitude layers is equivalent to Evans stability of the limiting constant layer, which in turn can be checked by a linear-algebraic computation. Finally, for a class of symmetric-dissipative systems including the physical examples mentioned above, we carry out energy estimates showing that constant (and thus small-amplitude) layers always satisfy uniform Evans stability. This yields existence of small-amplitude multi-dimensional boundary layers for the compressible Navier-Stokes and MHD equations. For both equations these appear to be the first such results in the compressible case.

## Contents



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# 1 Introduction

In this paper, we study existence and stability of noncharacteristic viscous boundary layers of hyperbolic–parabolic systems of the type arising in fluid and magnetohydrodynamics (MHD). In a companion paper [GMWZ6], we have shown under mild structural assumptions that for such layers, maximal linearized stability estimates, transversality of layer profiles, and satisfaction of the uniform Lopatinski condition by the associated residual hyperbolic system all follow from a uniform spectral stability condition on layer profiles that is expressible in terms of an Evans function (uniform Evans stability, Definition 3.4). Here we use these abstract results to obtain existence and stability in interesting physical applications. Our structural hypotheses are general enough to allow van der Waals equations of state.

The main results of this paper are as follows: (i) assuming the uniform Lopatinski condition and transversality of layer profiles (Definitions 1.16 and 1.11), we construct arbitrarily high-order approximate boundary-layer solutions matching an inner boundary-layer profile to an outer hyperbolic solution; (ii) assuming uniform Evans stability, we use results of [GMWZ4, GMWZ6] to show existence and stability of exact boundary-layer solutions close to the approximate solutions, and consequently we obtain convergence of viscous solutions to solutions of the residual hyperbolic problem in the small viscosity limit; (iii) we show that uniform Evans stability of small-amplitude boundary layers is equivalent to uniform Evans stability of the associated limiting constant layer; and (iv), we use (iii) to verify the uniform Evans condition for small amplitude layers for a class of *symmetric-dissipative systems* that includes the above physical examples as well as the class introduced by Rousset [R3].

In connection with (iii) and (iv) above, we prove existence of small-amplitude layer profiles for a variety of boundary conditions, including mixed Dirichlet-Neumann conditions. These profiles appear in the leading term of the approximate solutions referred to in (i).

The above results yield existence and stability of multi-dimensional small-amplitude noncharacteristic boundary-layer solutions of the compressible Navier-Stokes and MHD equations. In both cases these appear to be the first such results. For large-amplitude layers, the questions of existence and stability are reduced to verification of the uniform Evans condition. Efficient numerical methods for such verification are presented, for example, in [CHNZ, HLyZ2, HLyZ2].

Although most of the important physical examples can be written in conservative form, the general theory is presented for the more general nonconservative case in sections 2, 3, 4 and the appendices. As we noted in [GMWZ6, GMWZ7], the theory is clearer and in many ways simpler in the more general setting.

For results on stability of boundary layers for the incompressible Navier-Stokes equations we refer to [TW, IS].

### 1.1 Equations and assumptions

Consider as in [GMWZ6] a quasilinear hyperbolic–parabolic system

$$\mathcal{L}_\varepsilon(u) := A_0(u)u_t + \sum_{j=1}^d A_j(u)\partial_j(u) - \varepsilon \sum_{j,k=1}^d \partial_j\big(B_{jk}(u)\partial_k u\big) = 0, \tag{1.1}$$

on $[-T,T]\times\Omega$, where $\Omega\subset\mathbb{R}^d$ is an open set. We assume the block structure

$$A_0(u) = \begin{pmatrix} A_0^{11} & 0 \\ A_0^{21} & A_0^{22} \end{pmatrix}, \quad B_{jk}(u) = \begin{pmatrix} 0 & 0 \\ 0 & B_{jk}^{22} \end{pmatrix}, \tag{1.2}$$

a corresponding splitting

$$u = (u^1, u^2) \in \mathbb{R}^{N-N'} \times \mathbb{R}^{N'}, \tag{1.3}$$

and decoupled boundary conditions

$$\begin{cases} \Upsilon_1(u^1)_{|x\in\partial\Omega} = g_1(t,x), \\ \Upsilon_2(u^2)_{|x\in\partial\Omega} = g_2(t,x), \\ \Upsilon_3(u, \partial_T u^2, \partial_\nu u^2)_{|x\in\partial\Omega} = 0, \end{cases} \tag{1.4}$$

where $\partial_T$ and $\partial_\nu$ denote tangential and inward normal derivatives with respect to $\partial\Omega$ and $\Upsilon_3(u,\partial_T u^2,\partial_\nu u^2) = K_\nu\partial_\nu u^2 + \sum_{j=1}^p K_j(u)V_j u^2$, where the $V_j(x)$ are smooth vector fields tangent to $\partial\Omega$ and $K_\nu$ is constant. Unless otherwise noted we take $\Omega$ bounded with smooth (that is, $C^k$ for $k$ large) boundary, but our results apply with no essential change to other situations such as the case where $\partial\Omega$ coincides with a half-space outside a compact set.

Setting $\varepsilon = 0$ in (1.1) we obtain $\mathcal{L}_0$, a first-order operator assumed to hyperbolic. The parameter $\varepsilon$ plays the role of a non-dimensional viscosity and for $\varepsilon > 0$, the system is assumed to be parabolic or at least partially parabolic. Classical examples are the Navier-Stokes equations of gas dynamics and the equations of magneto-hydrodynamics (MHD).

We set

$$\overline{A}_j = A_0^{-1}A_j, \quad \overline{B}_{jk} = A_0^{-1}B_{jk}, \tag{1.5}$$

$$\overline{A}(u,\xi) = \sum_{j=1}^d \xi_j\overline{A}_j(u) \quad \text{and} \quad \overline{B}(u,\xi) = \sum_{j,k=1}^d \xi_j\xi_k\overline{B}_{jk}(u), \tag{1.6}$$

and systematically use the notation $M^{\alpha\beta}$ for the sub-blocks of a matrix $M$ corresponding to the splitting $u=(u^1,u^2)$. Note that

$$\overline{B}_{j,k}(u) := A_0(u)^{-1}B_{jk}(u) = \begin{pmatrix} 0 & 0 \\ 0 & \overline{B}_{jk}^{22}(u), \end{pmatrix}, \tag{1.7}$$

so it is natural to define the *high-frequency principal part* of (1.1) by

$$\begin{cases} \partial_t u^1 + \overline{A}^{11}(u,\partial)u^1 = 0, \\ \partial_t u^2 - \varepsilon \overline{B}^{22}(u,\partial)u^2 = 0. \end{cases} \tag{1.8}$$

Our main structural assumptions are modeled on the fundamental example of the Navier-Stokes equations with general, possibly van der Waals type equation of state. For applications it is important to allow the states assumed by solutions of (1.1) when $\varepsilon > 0$ or when $\varepsilon = 0$ to vary in overlapping but not necessarily identical regions of state space. Moreover, these regions must be allowed to depend on $(t,x)$. These considerations motivate the following definition of $\mathcal{U}$, $\mathcal{U}_\partial$, and $\mathcal{U}^*$.

For some $T > 0$ let $\mathcal{O}(t,x)$ be a continuous set-valued function from $[-T,T] \times \Omega$ to open sets in $\mathbb{R}^N$, and define graphs

$$\begin{aligned} &\mathcal{U} = \{(t,x,\mathcal{O}(t,x)) : (t,x) \in [-T,T] \times \Omega\} \\ &\mathcal{U}_\partial = \{(t,x_0,\mathcal{O}(t,x_0)) : (t,x_0) \in [-T,T] \times \partial\Omega\}. \end{aligned} \tag{1.9}$$

For $(t,x_0) \in [-T,T] \times \partial\Omega$ let $\mathcal{O}^*(t,x_0)$ be another continuous open-set-valued function satisfying $\mathcal{O}^*(t,x_0) \supset \mathcal{O}(t,x_0)$ and define

$$\mathcal{U}^* = \{(t,x_0,\mathcal{O}^*(t,x_0)) : (t,x_0) \in [-T,T] \times \partial\Omega\}. \tag{1.10}$$

Observe that we have

$$\mathcal{U}_\partial \subset \mathcal{U} \cap \mathcal{U}^*, \tag{1.11}$$

but neither $\mathcal{U}$ nor $\mathcal{U}^*$ is a subset of the other. For elements of $\mathcal{U}$ define

$$\pi(t,x,\mathcal{O}(t,x)) = \mathcal{O}(t,x) \tag{1.12}$$

and define $\pi$ similarly for elements of $\mathcal{U}_\partial$ and $\mathcal{U}^*$. Finally, denote by $\pi\mathcal{U}$ (resp. $\pi\mathcal{U}^*$, $\pi\mathcal{U}_\partial$) the union of the open sets obtained by applying $\pi$ to elements of $\mathcal{U}$ (resp. $\mathcal{U}^*$, $\mathcal{U}_\partial$).

The set $\pi\mathcal{U}$ is the "hyperbolic set" where solutions of the inviscid equation $\mathcal{L}_0(u) = 0$ take their values; $\pi\mathcal{U}^*$ is the set where boundary layer solutions $u^\varepsilon$ of the viscous equations, restricted to a small neighborhood of the boundary, take their values. In particular, the layer profiles (Definition 1.14) take values in $\pi\mathcal{U}^*$. The set $\pi\mathcal{U}_\partial \subset \pi\mathcal{U} \cap \pi\mathcal{U}^*$ is the set of profile endstates where matching of the two types of solutions occurs; more precisely, it contains the limits as $z \to \infty$ of layer profiles (see (1.15)), or equivalently, the boundary values of solutions of the associated residual hyperbolic problem (see 1.27).

**Assumptions 1.1.**

(H1) *The matrices $A_j$ and $B_{jk}$ are $C^\infty$ $N \times N$ real matrix-valued funtions of the variable $u \in \pi\mathcal{U} \cup \pi\mathcal{U}^* \subset \mathbb{R}^N$. Moreover, for all $u \in \pi\mathcal{U} \cup \pi\mathcal{U}^*$,* $\det A_0(u) \neq 0$.

(H2) *There is $c > 0$ such that for all $u \in \pi\mathcal{U} \cup \pi\mathcal{U}^*$ and $\xi \in \mathbb{R}^d$, the eigenvalues of $\overline{B}^{22}(u,\xi)$ satisfy* $\operatorname{Re}\mu \geq c|\xi|^2$.

(H3) *For all* $u \in \pi\mathcal{U} \cup \pi\mathcal{U}^*$ *and* $\xi \in \mathbb{R}^d \backslash \{0\}$, *the eigenvalues of* $\overline{A}^{11}(u,\xi)$ *are real, semi-simple and of constant multiplicity. Moreover, for* $u \in \pi\mathcal{U}^*$, $\det \overline{A}^{11}(u,\nu) \neq 0$, *with the eigenvalues of the normal matrix* $\overline{A}^{11}(u,\nu) \neq 0$ *all positive (inflow) or all negative (outflow), where* $\nu$ *denotes the inward normal to* $\partial\Omega$.

(H4) *For all* $u \in \pi\mathcal{U}$ *and* $\xi \in \mathbb{R}^d \backslash \{0\}$, *the eigenvalues of* $\overline{A}(u,\xi)$ *are real, semisimple, and of constant multiplicity. Moreover, for* $u \in \pi\mathcal{U}_\partial$, $\det \overline{A}(u,\nu) \neq 0$, *with number of positive (negative) eigenvalues of* $\overline{A}(u,\nu)$ *independent of* $u$.

(H5) *There is* $c > 0$ *such that for* $u \in \pi\mathcal{U}$ *and* $\xi \in \mathbb{R}^d$, *the eigenvalues of* $i\overline{A}(u,\xi)+\overline{B}(u,\xi)$ *satisfy* $\operatorname{Re}\mu \geq c\frac{|\xi|^2}{1+|\xi|^2}$.

**Remark 1.2.** *1.)In Hypothesis (H4) the statement "for* $u \in \mathcal{U}_\partial$, $\det \overline{A}(u,\nu) \neq 0$*" should be interpreted as asserting that for* $(t,x_0,u) \in \mathcal{U}_\partial$, *we have* $\det \overline{A}(u,\nu(x_0)) \neq 0$. *A similar remark applies to (H3) and to later statements of this sort.*

*2.) Hypothesis (H4) is a hyperbolicity condition on the inviscid equation* $\mathcal{L}_0(u) = 0$, *while* (H2),(H4) *implies hyperbolic–parabolicity of the viscous equation* $\mathcal{L}_\varepsilon(u) = 0$ *when* $\varepsilon > 0$. *(H3) is a hyperbolicity condition on the first equation in* (1.8). *The conditions on the normal matrices in* (H3)–(H4) *mean that the boundary is* noncharacteristic *for both the inviscid and the viscous equations. Hypothesis* (H5) *is a dissipativity condition reflecting genuine coupling of hyperbolic and parabolic parts for* $u \in \pi\mathcal{U}$.

*3.)Later we will occasionally drop the* $\pi$ *on* $\pi\mathcal{U}$ *in statements like* $u \in \mathcal{U}$ *below.*

Symmetry plays an important role in applications such as those to the Navier-Stokes and MHD equations considered here. In particular, (H5) holds always when the conditions in the following two definitions are satisfied [KaS1, KaS2].

**Definition 1.3.** *The system* (1.1) *is said to be symmetric dissipative if there exists a real matrix* $S(u)$, *which depends smoothly on* $u \in \pi\mathcal{U}$, *such that for all* $u \in \pi\mathcal{U}$ *and all* $\xi \in \mathbb{R}^d \backslash \{0\}$, *the matrix* $S(u)A_0(u)$ *is symmetric definite positive and block-diagonal,* $S(u)A(u,\xi)$ *is symmetric, and the symmetric matrix* $\operatorname{Re} S(u)B(u,\xi)$ *is nonnegative with kernel of dimension* $N - N'$.

Given that $A_0$ and the matrices $B_{jk}$ have the structure (1.2), observe that we have the equivalences:

$SA_0$ block diagonal $\Leftrightarrow$ $S$ lower block triangular $\Leftrightarrow$ $SB$ block diagonal.[1]

**Definition 1.4.** *A symmetric–dissipative system satisfies the genuine coupling condition if for all* $u \in \pi\mathcal{U}$ *and all* $\xi \in \mathbb{R}^d \backslash \{0\}$, *no eigenvector of* $\sum_j \overline{A}_j\xi_j$ *lies in the kernel of* $\sum_{j,k} \overline{B}_{jk}\xi_j\xi_k$.

The constant multiplicity condition in Hypothesis (H4) holds for the compressible Navier–Stokes equations whenever $\overline{A}(u,\xi)$ is hyperbolic. We are able to treat symmetric-dissipative systems like the equations of viscous MHD, for which the constant multiplicity condition fails, under the following relaxed hypothesis.

[1] The block-diagonal assumption repairs a minor omission in [GMWZ6]; this is needed to conclude (H5) from symmetric dissipativity plus the genuine coupling condition.

**Hypothesis H4$'$.** For all $u \in \pi\mathcal{U}$ and $\xi \in \mathbb{R}^d \setminus \{0\}$, the eigenvalues of $\overline{A}(u,\xi)$ are real and are either semisimple and of constant multiplicity or are totally nonglancing in the sense of [GMWZ6], Definition 4.3. Moreover, for $u \in \pi\mathcal{U}_\partial$ we have $\det \overline{A}(u,\nu) \neq 0$, with the number of positive (negative) eigenvalues of $\overline{A}(u,\nu)$ independent of $u$.

**Remark 1.5.** *The condition of constant multiplicity in (H3) can probably be dropped for symmetric–dissipative systems. For sufficiently small-amplitude boundary layers, we expect that the condition in (H3) that eigenvalues have a common sign may be dropped as well; see Remark 4.2.2.*

**Notations 1.6.** With assumptions as above, $N_+$ (constant) denotes the number of positive eigenvalues of $\overline{A}_\nu(u) := \overline{A}(u,\nu)$ for $u \in \pi\mathcal{U}_\partial$ and $N_+^1$ the number of positive eigenvalues of $\overline{A}_\nu^{11}(u) := \overline{A}^{11}(u,\nu)$ for $u \in \pi\mathcal{U}^*$. We also set $N_b = N' + N_+^1$.

As indicated by block structure (1.8), $N_b$ is the correct number of boundary conditions for the well posedness of (1.1), for solutions with values in $\mathcal{U}^* \cup \mathcal{U}$, with $N'$ boundary conditions for $u^2$ and $N_+^1$ boundary conditions for $u^1$. On the other hand, $N_+$ is the correct number of boundary conditions for the inviscid equation for solutions with values in $\mathcal{U}$.

**Assumption 1.7.** (H6)$\Upsilon_1$, $\Upsilon_2$ *and* $\Upsilon_3$ *are smooth functions of their arguments with values in* $\mathbb{R}^{N_+^1}$, $\mathbb{R}^{N'-N''}$ *and* $\mathbb{R}^{N''}$ *respectively, where* $N'' \in \{0,1,\dots,N'\}$. *Moreover,* $K_N$ *has maximal rank* $N''$ *and for all* $u \in \mathcal{U}^*$ *the Jacobian matrices* $\Upsilon_1'(u^1)$ *and* $\Upsilon_2'(u^2)$ *have maximal rank* $N_+^1$ *and* $N' - N''$ *respectively.*

**Examples 1.8.** Hypotheses (H1)–(H5) are satisfied by the compressible Navier–Stokes equations, provided the normal velocity of the fluid is nonvanishing on $\mathcal{U}^*$ and the normal characteristic speeds (eigenvalues of $\overline{A}(u,\nu)$) are nonvanishing on $\mathcal{U}_\partial$. We have $N_+^1 = 1$ or 0 according as normal velocity is positive with respect to inward normal (inflow) or negative (outflow). Recalling that $\mathcal{U}^*$ is the set where boundary layer solutions of the viscous equations, restricted to a small neighborhood of the boundary, take their values, we see that these velocity restrictions correspond to having a porous boundary through which fluid is pumped in or out, in contrast to the characteristic, no-flux boundary conditions encountered at a solid material interface for which normal velocity is set to zero.

Hypotheses (H1)-(H5), with (H4) replaced by (H4$'$), are satisfied by the viscous MHD equations with ideal gas equation of state under similar velocity restrictions on the plasma. See Remark 5.6 and the discussion following it for the precise velocity requirements and for examples of boundary conditions for viscous MHD and the corresponding reduced hyperbolic problem.

We note that our structural hypotheses are sometimes satisfied under much weaker assumptions on the equation of state, which may be of van der Waals type on $\mathcal{U}^*$ and just thermodynamically stable on $\mathcal{U}$; see [GMWZ4, Z3].

Boundary conditions of the type we consider for the compressible Navier-Stokes equations are of considerable practical importance. Recall the equations are

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u^t u) + \nabla p = \varepsilon\mu\Delta u + \varepsilon(\mu+\eta)\nabla \operatorname{div} u \\ \partial_t(\rho E) + \operatorname{div}\big((\rho E + p)u\big) = \kappa\Delta T + \varepsilon\mu \operatorname{div}\big((u\cdot\nabla)u\big) \\ \qquad\qquad\qquad\qquad\qquad\qquad + \varepsilon(\mu+\eta)\nabla(u\cdot \operatorname{div} u) \end{cases} \tag{1.13}$$

where $\rho$ denotes density, $u$ velocity, $e$ specific internal energy, $E = e + \frac{|u|^2}{2}$ specific total energy, $p = p(\rho, e)$ pressure, and $T = T(\rho, e)$ temperature. Take the unknowns to be $(\rho, u, T)$, and consider the problem on the exterior $\Omega = \alpha^c$ of a bounded set $\alpha$ with smooth boundary, with no-slip *suction-type* boundary conditions on the velocity,

$$u_T|_{\partial\Omega} = 0, \quad u_\nu|_{\partial\Omega} = V(x) < 0,$$

and either prescribed or insulative boundary conditions on the temperature,

$$T|_{\partial\Omega} = T_{wall}(x) \quad \text{or} \quad \partial_\nu T|_{\partial\Omega} = 0.$$

This corresponds to the situation of an airfoil with microscopic holes through which gas is pumped from the surrounding flow, the microscopic suction imposing a fixed normal velocity while the macroscopic surface imposes standard temperature conditions as in flow past a (nonporous) plate. This configuration was suggested by Prandtl and tested experimentally by G.I. Taylor as a means to reduce drag by stabilizing laminar flow; see [S, Br]. It was implemented in the NASA F-16XL experimental aircraft program in the 1990's with reported 25% reduction in drag at supersonic speeds [Br].[2]

**Remark 1.9.** *At the expense of further bookkeeping, we could equally well define separate $\mathcal{U}_{\partial,j} \subset \mathcal{U}^*_j$ for each connected component $(\partial\Omega)_j$ of the boundary $\partial\Omega$, each with distinct values of $N_+$, $N^1_+$. where now the sets $\mathcal{U}_{\partial,j}$ are to be thought of as the sets of possible boundary values for the hyperbolic problem at each $(\partial\Omega)_j$. This would allow, for example, the situation that fluid is pumped in through one boundary and out another as considered in [TW] for the incompressible case.*

We follow hypotheses (H1)–(H6), in some cases with (H4′) in place of (H4), throughout the paper; *unless otherwise indicated, they are assumed in all statements and propositions that follow.*

## 1.2 Layer profiles, transversality, and $\mathcal{C}$ manifolds

To match solutions of the inviscid problem approaching a constant value $\underline{u}$ at $x_0 \in \partial\Omega$ to solutions satisfying the hyperbolic–parabolic boundary conditions, one looks for exact solutions of (1.1) (1.4) on the half-space $(x - x_0)\cdot\nu(x_0) \geq 0$ tangent to $\partial\Omega$ at $x_0$ of the form

$$u_\varepsilon(t,x) = w\Big(\frac{(x-x_0)\cdot\nu}{\varepsilon}\Big), \tag{1.14}$$

[2] See also NASA site http://www.dfrc.nasa.gov/Gallery/photo/F-16XL2/index.html

$\nu(x_0)$ the inward unit normal to $\partial\Omega$ at $x_0$, such that

$$\lim_{z\to+\infty} w(z) = \underline{u}\,. \tag{1.15}$$

Solutions are called *layer profiles.*

The equation for $w$ reads

$$\begin{cases} A_\nu(w)\partial_z w - \partial_z\big(B_\nu(w)\partial_z w\big) = 0, & z \geq 0, \\ \Upsilon(w, 0, \partial_z w^2)_{|z=0} = (g_1(t,x_0), g_2(t,x_0), 0), \end{cases} \tag{1.16}$$

where $A_\nu(u) = \sum_{j=1}^d A_j(u)\nu_j$ and $B_\nu(u) := \sum B_{jk}(u)\nu_j\nu_k$. The natural limiting boundary conditions for the inviscid problem should be satisfied precisely by those states $\underline{u}$ that are the endstates of layer profiles. This leads us to define

$$\mathcal{C}(t,x_0) = \{\underline{u} : \text{ there is a layer profile } w \in C^\infty(\overline{\mathbb{R}}_+;\mathcal{U}^*) \text{ satisfying (1.15), (1.16) }. \tag{1.17}$$

The profile equation (1.16) can be written as a first order system for $U = (w, \partial_z w^2)$, which is nonsingular if and only if $A_\nu^{11}$ is invertible, (H3):

$$\begin{aligned} \partial_z w^1 &= -(A_\nu^{11})^{-1}A_\nu^{12} w^3, \\ \partial_z w^2 &= w^3, \\ \partial_z\big(B_\nu^{22} w^3\big) &= \big(A_\nu^{22} - A_\nu^{21}(A_\nu^{11})^{-1}A_\nu^{12}\big)w^3, \end{aligned} \tag{1.18}$$

and the matrices are evaluated at $w = (w^1, w^2)$.

Consider now the linearized equations of (1.16) about $w(z)$, written as a first-order system

$$\partial_z \dot W - \mathcal{G}_\nu(z)\dot W = 0, \qquad z \geq 0, \tag{1.19}$$

$$\Gamma_\nu \dot W_{|z=0} = 0 \tag{1.20}$$

in $\dot W = (\dot w^1, \dot w^2, \dot w^3)$, where

$$\mathcal{G}_\nu(+\infty) := \lim_{z\to+\infty}\mathcal{G}_\nu(z) = \begin{pmatrix} 0 & 0 & -(A_\nu^{11})^{-1}A_\nu^{12} \\ 0 & 0 & I \\ 0 & 0 & (B_\nu^{22})^{-1}(A_\nu^{22} - A_\nu^{21}(A_\nu^{11})^{-1}A_\nu^{12}) \end{pmatrix}(\underline{u}) \tag{1.21}$$

and (note decoupling between $u^1$ and $(u^2, u^3) := (u^2, \partial_\nu u^2)$ variables)

$$\Gamma_\nu U = \big(\Gamma_1 u^1, \Gamma_2 u^2, K_\nu u^3\big). \tag{1.22}$$

**Lemma 1.10** ([GMWZ6, MaZ3]). *Let $N_-^2$ denote the number of stable eigenvalues $\Re\mu < 0$ of $\mathcal{G}_\nu(+\infty)$, $N_+^2$ the number of unstable eigenvalues $\Re\mu > 0$, $\mathcal{S}$ the subspace of solutions of* (1.19) *that approach finite limits as $z \to \infty$, and $\mathcal{S}_0$ the subspace of solutions of* (1.19) *that decay to* 0*. Then,*

*(i) $N_-^2 + N_+^2 = N'$ and*

$$N_+ + N_-^2 = N_b := N' + N_+^1, \tag{1.23}$$

*(ii) profile $w(\cdot)$ decays exponentially to its limit $\underline{u}$ as $z \to +\infty$ in all derivatives, and*

*(iii)* $\dim \mathcal{S} = N + N_-^2$ *and* $\dim \mathcal{S}_0 = N_-^2$*.*

*Proof.* A proof is given in [GMWZ6], Lemma 2.12. □

**Definition 1.11.** *The profile $w$ is said to be* transversal *if*

*i) there is no nontrivial solution $\dot{w} \in \mathcal{S}_0$ which satisfies the boundary conditions $\Gamma_\nu(\dot{w}, \partial_z \dot{w}^2)_{|z=0} = 0$,*

*ii) the mapping $\dot{w} \mapsto \Gamma_\nu(\dot{w}, \partial_z \dot{w}^2)_{|z=0}$ from $\mathcal{S}$ to $\mathbb{C}^{N_b}$ has rank $N_b$.*

Definition 1.11(i) and (ii) corresponds to the geometric conditions that the level set $\{W : \Upsilon(W) = \Upsilon(w(0), 0, \partial_z w^2(0))\}$ have transversal intersections in phase space $W = (w, \partial_z w^2)$ at $W_0 := (w(0), \partial_z w(0))$ with the stable and center–stable manifolds, respectively, of $W_\infty := (\underline{u}, 0)$; see Lemma 5.3 below.

The following assumption is the starting point for our construction of exact boundary layer solutions to (1.1).

**Assumption 1.12.** *Fix a choice of $(g_1, g_2)$ as in* (1.4)*. For $\mathcal{U}_\partial$ as in* (1.9) *we are given a smooth manifold $\mathcal{C}$ defined as the graph*

$$\mathcal{C} = \{(t, x_0, \mathcal{C}(t, x_0)) : (t, x_0) \in [-T, T] \times \partial\Omega\} \subset \mathcal{U}_\partial, \tag{1.24}$$

*where each $\mathcal{C}(t, x_0)$, defined as in* (1.17)*, is now assumed to be a smooth manifold of dimension $N - N_+$ . In addition we are given a smooth function*

$$W : [0, \infty) \times \mathcal{C} \to \pi\mathcal{U}^* \tag{1.25}$$

*such that for all $(t, x_0, q) \in \mathcal{C}$, $W(z, t, x_0, q)$ is a transversal layer profile satisfying* (1.16) *with $\nu = \nu(t, x_0)$ and converging to $q$ as $z \to \infty$ at an exponential rate that can be taken uniform on compact subsets of $\mathcal{C}$.*

This assumption is hard to check in general. However, in Proposition 2.6 we show that for a large class of problems including the Navier-Stokes and MHD equations with various boundary conditions, Assumption 1.12 is always satisfied for small-amplitude profiles. In Proposition 2.4 we give necessary and sufficient conditions on boundary operators of the form (1.4) in order for Assumption 1.12 to hold in the small-amplitude case. These boundary conditions include the standard noncharacteristic boundary conditions for the Navier-Stokes and viscous MHD equations. In Proposition 2.8 we give a *local* construction of a $\mathcal{C}$ manifold with associated profiles near a given, possibly large amplitude, transversal profile. In Proposition 2.10 we give a global construction of a $\mathcal{C}$ manifold with associated profiles near a given family of large-amplitude profiles, assuming such a family exists (not clear in general).

As regards the failure of Assumption 1.12, Proposition 2.6 implies, for example, the following nontransversality result.

**Proposition 1.13.** *Whenever* $\text{rank}\Upsilon_3 = N'' > N_-^2$*, small-amplitude profiles are not transversal. Equivalently, whenever the number of scalar Dirichlet conditions for the parabolic problem,* $(N' - N'') + N_+^1$*, is strictly less than the number of scalar boundary conditions for the residual hyperbolic problem,* $N_+$ *(see* (1.27)*), small-amplitude profiles are not transversal.*

## 1.3 Inviscid solutions and the uniform Lopatinski condition.

Under suitable assumptions we will study the small viscosity limit of solutions to

$$\begin{aligned}
&\mathcal{L}_\varepsilon(u) := A_0(u)u_t + \sum_{j=1}^{d} A_j(u)\partial_j u - \varepsilon \sum_{j,k=1}^{d} \partial_j\big(B_{jk}(u)\partial_k u\big) = 0, \\
&\Upsilon(u, \partial_T u^2, \partial_\nu u^2) = (g_1, g_2, 0) \text{ on } [0, T_0] \times \partial\Omega
\end{aligned} \tag{1.26}$$

and demonstrate convergence to a solution $u^0(t, x)$ of the inviscid hyperbolic problem:

$$\begin{aligned}
&\mathcal{L}_0(u^0) = 0 \text{ on } [0, T_0] \times \Omega \\
&u^0(t, x_0) \in \mathcal{C}(t, x_0) \text{ for } (t, x_0) \in [0, T_0] \times \partial\Omega,
\end{aligned} \tag{1.27}$$

where $\mathcal{C}(t, x_0)$ is the endstate manifold defined in Assumption 1.12 (see also Prop. 2.6).

**Definition 1.14.** *We refer to the boundary condition in* (1.27) *as the* residual hyperbolic boundary condition.

For $(t, x_0) \in [0, T_0] \times \partial\Omega$ we freeze a state $p := u^0(t, x_0)$ and, working in coordinates where the boundary is $x_d = 0$, we define

$$H(p, \zeta) := -A_d(p)^{-1}\left((i\tau + \gamma)A_0(p) + \sum_{j=1}^{d-1} i\eta_j A_j(p)\right). \tag{1.28}$$

Here we have suppressed the dependence of the frozen $A_j$ on spatial coordinates in the notation. Let

$$\psi : \mathbb{R}^N \to \mathbb{R}^{N_+} \tag{1.29}$$

be a defining function for $\mathcal{C}(t, x_0)$ near $p$, i.e., $\mathcal{C}(t, x_0) = \{u : \psi(u) = 0\}$, with $\nabla\psi$ full rank $N_+$. Then, the residual boundary condition (1.27) may be expressed, locally to $p$, as $\Upsilon_{res}(u) := \psi(u)$, hence the *linearized residual boundary condition* at $p$ takes the form

$$\Gamma_{res}(p)\dot{u} = 0 \Leftrightarrow \psi'(p)\dot{u} = 0 \Leftrightarrow \dot{u} \in T_p\mathcal{C}(t, x_0). \tag{1.30}$$

**Remark 1.15.** *Suppose $w(z)$ is a solution of* (1.16) *converging to $p = u^0(t, x_0) \in \mathcal{C}(t, x_0)$ as $z \to \infty$. Let us write the linearized equations of* (1.16) *around $w(z)$ as*

$$\mathbb{L}(t, x_0, z, \partial_z)\dot{w} = 0, \;\; \Gamma_{\nu(x_0)}(\dot{w}, \dot{w}_z^2) = 0. \tag{1.31}$$

*Observe that the tangent space $T_p\mathcal{C}(t, x_0)$ may be characterized as the set of limits at $z = \infty$ of solutions to* (1.31)*. This follows readily from the definition of $\mathcal{C}(t, x_0)$; see [Met4], Prop. 5.5.5.*

**Definition 1.16.** *1) The inviscid problem* (1.27) *satisfies the* uniform Lopatinski condition at $p = u(t, x_0)$ *provided there exists $C > 0$ such that for all $\zeta$ with $\gamma > 0$*

$$|D_{Lop}(p, \zeta)| := |\det\big(\mathbb{E}^-(H(p, \zeta)), \ker\Gamma_{res}(p)\big)| \geq C. \tag{1.32}$$

*2) The inviscid problem* (1.27) *satisfies the* uniform Lopatinski condition *provided* (1.32) *holds with a constant that can be chosen independently of $(t, x_0) \in [0, T_0] \times \partial\Omega$.*

Here by a determinant of subspaces we mean the determinant of the matrix with subspaces replaced by smoothly chosen bases of column vectors, specifying $D_{Lop}$ up to a smooth nonvanishing factor; for example, if the bases are taken to be orthonormal, then $|D_{Lop}|$ is independent of the choice of bases. We refer to [GMWZ7], section 4.1 for equivalent formulations and further discussion of the uniform Lopatinski condition.

**Theorem 1.17.** *Given a smooth manifold $\mathcal{C}$ as in Assumption 1.12, consider the hyperbolic problem* (1.27)

*(i) under hypotheses (H1)-(H5), or alternatively,*

*(ii) assuming (H1)-(H5), except that (H4) is replaced by (H4') in the symmetric-dissipative case.*

*Let $s > \frac{d}{2} + 1$ and suppose that we are given initial data $v^0(x) \in H^{s+1}(\Omega)$ at $t = 0$ satisfying corner compatibility conditions to order $s - 1$ for* (1.27)*. Suppose also that the uniform Lopatinski condition is satisfied at all points $x_0 \in \partial\Omega$, $t = 0$. Then there exists a $T_0 > 0$ and a function $u^0(t, x) \in H^s([0, T_0] \times \Omega)$ satisfying* (1.27) *with*

$$u^0_{|t=0} = v^0, \tag{1.33}$$

*and so that the uniform Lopatinski condition holds on $[0, T_0] \times \partial\Omega$.*

*Proof.* We refer to [CP], Chapter 7 for a discussion of corner compatibility conditions. For the proof in the case of constant multiplicity (i.e., when (H4) holds) see [CP], Chapter 7 and [Met2]. For the case of variable multiplicities (including situations more general than those considered here) see [MZ2].

□

### 1.4 Approximate solutions to the viscous problem

The first step in constructing exact solutions to the viscous problem (1.26) that converge to a given solution $u^0$ of the hyperbolic problem (1.27) in the small viscosity limit is to construct high order approximate solutions of (1.26) with that property. Following the approach of [GMWZ4, GMWZ7] we construct approximate solutions using a WKB expansion as described in the following result. A more precise statement of Proposition 1.18 and the proof are given in Appendix A.

Let

$$(x_0, z) : \Omega \to \partial\Omega \times \mathbb{R}^+ \tag{1.34}$$

be a smooth map defined for $d(x, \partial\Omega) \leq r$, $r > 0$ sufficiently small, such that

$$(x_0, z)^{-1}(x_0, z) = x_0 + z\nu(x_0),$$

where $\nu(x_0)$ is the inward normal to $\partial\Omega$ at $x_0$, i.e., $(x_0, z)$ are normal coordinates and $\nabla z = \nu$ on $\partial\Omega$. Let $\chi(x)$ be a smooth cutoff function vanishing for $d(x, \partial\Omega) \geq 2r$ and identically one for $d(x, \partial\Omega) \leq r$.

**Proposition 1.18** (Approximate solutions)**.** *Suppose we are given a $\mathcal{C}$ manifold and associated transversal profiles $W(z, t, x_0, q)$ as in Assumption 1.12, and also a solution $u^0 \in H^{s_0}$ to the inviscid problem* (1.27) *as described in Theorem 1.17. (In particular, the uniform Lopatinski condition is satisfied on* $[0, T_0] \times \partial\Omega$*). Fix positive integers $M$ and $s$ with $M \geq s \geq 1$. Provided $s_0$ is sufficiently large relative to $M$ and $s$ (see* (A.25)*), there exists an approximate solution $u^\varepsilon_a(t, x)$ to the viscous problem* (1.26) *on* $[0, T_0] \times \Omega$ *of the form*

$$\begin{aligned} &u^\varepsilon_a(t, x) = \sum_{0\leq j\leq M} \varepsilon^j \mathcal{U}^j(t, x, \frac{z}{\epsilon}) + \epsilon^{M+1} u^M(t, x), \\ &\mathcal{U}^j(t, x, \frac{z}{\epsilon}) = \chi(x)\left(W^j(\frac{z}{\varepsilon}, t, x_0) - W^j(+\infty, t, x_0)\right) + u^j(t, x). \end{aligned} \tag{1.35}$$

*Here $u^0$ is the given inviscid solution, the leading profile $W^0$ is given by*

$$W^0(Z, t, x_0) := W(Z, t, x_0, u^0(t, x_0)) \tag{1.36}$$

*for $W(Z, t, x_0, q)$ as in Assumption 1.12, and the higher profiles $W^j(Z, t, x_0)$ converge exponentially to their limits as $Z \to +\infty$. The approximate solution $u^\varepsilon_a$ satisfies*

$$\begin{aligned} &\|\mathcal{L}_\varepsilon(u_a)\|_{H^s([0,T_0]\times\Omega)} \leq C\varepsilon^M \\ &\Upsilon(u_a, \partial_T u^2_a, \partial_\nu u^2_a) = (g_1, g_2, 0) \text{ on } [0, T_0] \times \partial\Omega \end{aligned} \tag{1.37}$$

*for $(g_1, g_2, 0)$ as in the original viscous problem* (1.26).

**Examples 1.19.** The combination of Theorem 1.17 (for inviscid solutions $u^0$), Proposition 2.6 (for $\mathcal{C}$ manifolds and transversal profiles), and Corollary 1.29 (for satisfaction of the uniform Evans condition which implies transversality and uniform Lopatinski) provides us with a variety of examples, involving both Dirichlet and Neumann conditions for the Navier Stokes and viscous MHD equations and including all cases mentioned in Examples 1.8, where the hypotheses of Proposition 1.18 are satisfied for small-amplitude profiles.

The large-amplitude case is more difficult. In section 5.1 we construct $\mathcal{C}$ manifolds with associated large-amplitude profiles for the isentropic Navier-Stokes equations. Transversality of those profiles is verified in Proposition 5.5 and maximal dissipativity of the residual hyperbolic problem, which implies the uniform Lopatinski condition, is verified in section 5.1.1. Thus, here again the hypotheses of Proposition 1.18 are satisfied.

Large-amplitude $\mathcal{C}$ manifolds and transversal profiles are constructed *locally* near the endstate of a *given* transversal profile in Proposition 2.8. In problems with rotational symmetry it may be possible to promote locally constructed large-amplitude $\mathcal{C}$ manifolds to global ones by using that symmetry. For large-amplitude layers there are now efficient numerical methods available for verification of the uniform Evans condition. [CHNZ, HLyZ2, HLyZ2]

## 1.5 The Evans condition, stability and transversality, and the small viscosity limit

We refer to the approximate solution $u_a^\varepsilon$ described in Proposition 1.18 as an approximate boundary-layer solution. Under a suitable Evans condition (Definition 1.20), we will produce a nearby *exact* boundary-layer solution $u^\varepsilon$ of (1.1), (1.4) of the form

$$u^\varepsilon = u_a^\varepsilon + v^\varepsilon, \tag{1.38}$$

where $v^\varepsilon$ satisfies the "error problem"

$$\mathcal{L}_\varepsilon(u_a + v) - \mathcal{L}_\varepsilon(u_a) = -\mathcal{L}_\varepsilon(u_a), \tag{1.39}$$

derived from the problems satisfied by $u^\varepsilon$ and $u_a^\varepsilon$, with boundary data

$$\big(\Upsilon(u_a + v, \partial_{T,\nu}(u_a^2 + v^2)) - \Upsilon(u_a, \partial_{T,\nu}u_a^2)\big)|_{x\in\partial\Omega} = 0 \tag{1.40}$$

and forcing term $-\mathcal{L}_\varepsilon(u_a)$ small of order $\varepsilon^M$ in $H^s$ as described in (1.37), with $M$ and $s$ large. Here $v_{|t=0}$ satisfies high order corner compatibility conditions depending on $u_a$.

Evidently, the problem of estimating convergence error $\|v^\varepsilon\|_{H^s}$ in terms of truncation error (1.39) amounts to determining $H^s \to H^s$ stability estimates for (1.39)–(1.40), for which the main obstacle is rapid variation of coefficients in the boundary-layer region. We focus now on this region, and stability of associated layer-profiles, i.e., the "microscopic" stability problem.

Fix a point $(t, x_0)$, $x_0 \in \partial\Omega$ and consider again the viscous problem (1.26). We work in local spatial coordinates $(y, x_d)$ where $x_0$ is $(0, 0)$ and $\partial\Omega$ is given by $x_d = 0$. Consider a planar layer profile

$$u^\varepsilon(t,y,x_d) = w(x_d/\varepsilon) \tag{1.41}$$

as in (1.14), which is an exact solution to (1.26) on $x_d \geq 0$ when the coefficients and boundary data $(g_1, g_2, 0)$ are frozen at $(t, x_0)$. Without loss of generality we take $\varepsilon = 1$, set $z = x_d$, and write the linearized equations of (1.26) about $w$:

$$\mathcal{L}'_w \dot{u} = \dot{f}, \quad \Upsilon'(\dot{u}, \partial_y \dot{u}^2, \partial_z \dot{u}^2)_{|x=0} = \dot{g}. \tag{1.42}$$

Here $\Upsilon'$ is the differential of $\Upsilon$ at $(w(0), 0, \partial_z w^2(0))$ and $\mathcal{L}'_w$ is a differential operator with coefficients that are smooth functions of $x_d$.

Performing a Laplace-Fourier transform of (1.42) in $(t, y)$, with frequency variables denoted by $\gamma + i\tau$ and $\eta$ respectively, yields the family of ordinary differential systems

$$L(z, \gamma + i\tau, i\eta, \partial_z)u = f, \quad \Upsilon'(u, i\eta u^2, \partial_z u^2)_{|z=0} = g, \tag{1.43}$$

$$L = -\mathcal{B}(z)\partial_z^2 + \mathcal{A}(z,\zeta)\partial_z + \mathcal{M}(z,\zeta), \tag{1.44}$$

with in particular, $\mathcal{B}(z) = B_{d,d}(w(z))$ and $\mathcal{A}^{11}(z,\zeta) = A_d^{11}(w(z))$. The matrices $\mathcal{A}(z,\zeta)$, $\mathcal{M}(z,\zeta)$ are written out explicitly in (B.30).

The problem (1.43) may be written as a first order system

$$\partial_z U - \mathcal{G}(z,\zeta)U = F, \quad \Gamma(\zeta)U_{|z=0} = G, \tag{1.45}$$

where $U = (u, \partial_z u^2) = (u^1, u^2, u^3) \in \mathbb{C}^{N+N'}$ and $\zeta = (\tau, \gamma, \eta)$. The components of $\mathcal{G}(z,\zeta)$ are given explicitly in (B.33) and we have

$$\Gamma(\zeta)U = (\Gamma^1 u^1, \Gamma^2 u^2, \Gamma^3(\zeta)u^3) \tag{1.46}$$

with

$$\begin{aligned} &\Gamma^1 u^1 = \Upsilon'_1(w^1(0))u^1, \ \Gamma^2 u^2 = \Upsilon'_2(w^2(0))u^2, \\ &\Gamma^3(\eta)U = K_d u^3 + K_T(w(0))i\eta u^2. \end{aligned} \tag{1.47}$$

Observe that when $\zeta = 0$ (1.45) coincides with (1.19)–(1.20) in the case when $\nu = (0,1)$.

A necessary condition for stability of the inhomogeneous equations (1.45) is stability of the homogeneous case $F = 0$, $G = 0$, i.e., nonexistence for $\gamma \geq 0$, $\zeta \neq 0$ of solutions $U$ decaying as $z \to +\infty$ and satisfying $\Gamma(\zeta)U(0) = 0$. These may be detected by vanishing of the *Evans function*

$$D(\zeta) := \det_{N+N'}(\mathbb{E}^-(\zeta), \ker\Gamma(\zeta)), \tag{1.48}$$

where $\mathbb{E}^-(\zeta)$ is the subspace of initial data at $z = 0$ for which the solution of $\partial_z U - \mathcal{G}(z,\zeta)U = 0$ decays at $z = +\infty$. For high frequencies $|\zeta| \geq R > 0$ we also define in (3.14) a rescaled Evans function $D^{sc}(\zeta)$.

Given a $\mathcal{C}$ manifold and associated layer profiles $W(Z, t, x_0, q)$ as in Assumption 1.12, along with an inviscid solution $u^0$ as in Theorem 1.17, we define in the same way $D(t, x_0, \zeta)$ and $D^{sc}(t, x_0, \zeta)$ for *every* $(t, x_0) \in [0, T_0] \times \partial\Omega$ using the associated profile $w(Z) = W(Z, t, x_0, u(t, x_0))$.

**Definition 1.20.** *1.) We say that the* uniform Evans condition is satisfied by the profile *$w(z)$ provided there exist positive constants $R$, $C$ such that*

$$\begin{aligned}&|D(\zeta)| \geq C \text{ for } 0 < |\zeta| \leq R,\ \gamma \geq 0 \text{ and}\\ &|D^{sc}(\zeta)| \geq C \text{ for } |\zeta| \geq R,\ \gamma \geq 0.\end{aligned} \tag{1.49}$$

*2.) We say that the* uniform Evans condition holds on $[0, T_0] \times \partial\Omega$ *provided there exist positive constants $R$, $C$ such that the estimates* (1.49) *hold for the Evans functions $D(t, x_0, \zeta)$ and $D^{sc}(t, x_0, \zeta)$ defined above, uniformly for all $(t, x_0) \in [0, T_0] \times \partial\Omega$.*

In order to understand the behavior of $D(\zeta)$ near $\zeta = 0$ it is helpful to introduce polar coordinates

$$\zeta = \rho\hat{\zeta},\ \rho = |\zeta|, \text{ for } \zeta \neq 0, \gamma \geq 0 \tag{1.50}$$

and to write

$$D(\zeta) = D(\hat{\zeta}, \rho), \quad \mathbb{E}^-(\zeta) = \mathbb{E}^-(\hat{\zeta}, \rho) \text{ for } \zeta \neq 0, \gamma \geq 0. \tag{1.51}$$

It is shown in [MZ3], Theorem 3.3 and [GMWZ6], Remark 2.31 that the spaces $\mathbb{E}^-(\zeta) = \mathbb{E}^-(\hat{\zeta}, \rho)$ have continuous extensions to $\rho = 0$ under our structural hypotheses (and, more generally, when there exist $K-$families of symmetrizers for the linearized viscous problem.) Hence, under our assumptions on $\Gamma$, $D(\hat{\zeta}, \rho)$ extends continuously to $\rho = 0$ for $\hat{\zeta}$ with $\hat{\gamma} \geq 0$. This continuity allows us to rephrase the low frequency uniform Evans condition equivalently as

$$D(\hat{\zeta}, 0) \neq 0 \text{ for } |\hat{\zeta}| = 1,\ \ \hat{\gamma} \geq 0. \tag{1.52}$$

The following elementary result allows us to verify Evans conditions by proving trace estimates.

**Lemma 1.21** ([GMWZ6], Lemma 2.19)**.** *Suppose that $\mathbb{E} \subset \mathbb{C}^n$ and $\Gamma : \mathbb{C}^n \to \mathbb{C}^m$, with* rank $\Gamma = \dim\mathbb{E} = m$. *If $|\det(\mathbb{E}, \ker\Gamma)| \geq c > 0$, then there is $C$, which depends only on $c$ and $|\Gamma^*(\Gamma\Gamma^*)^{-1}|$ such that*

$$|U| \leq C|\Gamma U| \textit{ for all } U \in \mathbb{E}. \tag{1.53}$$

*Conversely, if this estimate is satisfied then $|\det(\mathbb{E}, \ker\Gamma)| \geq c > 0$, where $c > 0$ depends only on $C$ and $|\Gamma|$.*

**Remark 1.22.** *By Lemma 1.21, the uniform Evans condition $|D(\zeta)| \geq C > 0$ on some subset $S$ of frequencies is equivalent to*

$$|U| \leq C|\Gamma U| \text{ for all } U \in \mathbb{E}^-(\zeta) \tag{1.54}$$

*for some constant $C > 0$ independent of $\zeta \in S$.*

We now recall two results from [GMWZ6]. The first extends an earlier result of Rousset [R2] and shows that only the low frequency Evans condition is needed for the construction of high order approximate viscous solutions $u^\varepsilon_a$.

**Lemma 1.23** ([GMWZ6] Theorem. 2.28)**.** *Assume (H1)-(H6) (with (H4′) replacing (H4) in the symmetric-dissipative case), and consider a layer profile $w(z) \to p$ as $z \to \infty$. The uniform Evans condition holds for low frequencies, that is, there exist positive constants $r$, $c$ such that*

$$|D(\zeta)| \geq c \text{ for } |\zeta| \leq r, \tag{1.55}$$

*if and only if $w$ is transversal and the uniform Lopatinski condition (Definition 1.16) holds at $p$ for the residual hyperbolic problem* (1.28)-(1.30)*.*

The next result shows that the full uniform Evans condition implies maximal linearized stability estimates for the viscous problem.

**Proposition 1.24** ([GMWZ6], Theorems 3.9 and 7.2)**.** *Assume (H1)-(H6) (with (H4′) replacing (H4) in the symmetric dissipative case) and consider the problem* (1.42) *defined by linearization around a layer profile $w(z)$. If the uniform Evans condition is satisfied by $w(z)$ (Definition 1.20), then solutions to* (1.42) *satisfy the maximal stability estimates* (3.16) *and* (3.8)*.*

Proposition 1.24 is proved by constructing smooth Kreiss-type symmetrizers for the Laplace–Fourier transformed equations (1.45) in the low-, medium-, and high-frequency regimes. This method of proof is just as important as the result, since the same symmetrizer construction may be used to obtain maximal stability estimates for the linearized equations about general (nonplanar) solutions of the form (1.35). The procedure, which is used in [MZ1, GMWZ3, GMWZ4], is to freeze slow variables in the original linearized viscous problem, take the Laplace-Fourier transform to obtain ODEs depending on frequency like (1.45), construct symmetrizers for the transformed problem, and then quantize those symbols to produce paradifferential operator symmetrizers for the original (unfrozen) problem. The resulting linear estimates can be used to prove convergence of a nonlinear iteration scheme that yields existence of an exact solution to the viscous problem (1.26) that is close to the approximate solution $u^\varepsilon_a$, as stated in the following theorem.

Collecting these observations and combining with Lemma 1.23 and Theorem 1.18, we obtain the following main result, which reduces the problem of proving existence and nonlinear stability of boundary-layer solutions to verification of the uniform Evans condition.

**Theorem 1.25.** *Consider the viscous problem* (1.26) *under assumptions (H1)-(H6) (or with (H4′) replacing (H4) in the symmetric-dissipative case). Given an inviscid solution $u^0 \in H^{s_0}([0,T_0] \times \Omega)$ as in Theorem 1.17, suppose that the uniform Evans condition holds on $[0,T_0] \times \partial\Omega$ (Defn. 1.20). Suppose the constants $k$, $M$, and $s_0$ satisfy*

$$k > \frac{d}{2} + 4, \quad M > k + 2, \quad s_0 > k + \frac{7}{2} + 2M + \frac{d+1}{2}. \tag{1.56}$$

*Then there exists* $\varepsilon_0 > 0$*, an approximate solution* $u_a^\varepsilon$ *as in* (1.35) *satisfying* (1.37)*, and an exact solution* $u^\varepsilon$ *of* (1.26) *such that for* $0 < \varepsilon \le \varepsilon_0$

$$\begin{aligned} &\|u^\varepsilon - u_a^\varepsilon\|_{W^{1,\infty}([0,T_0]\times\Omega)} \le C\varepsilon^{M-k}, \\ &\|u - u^0\|_{L^2(\Omega\times[0,T])} \le C\varepsilon^{1/2}, \\ &u^\varepsilon - u^0 = O(\varepsilon) \ \mathit{in}\ L^\infty_{loc}([0,T_0]\times\Omega^\circ) \end{aligned} \tag{1.57}$$

*where* $\Omega^\circ$ *denotes the interior of* $\Omega$*. Moreover, the linearized equations about either* $u_a^\varepsilon$ *or* $u^\varepsilon$ *satisfy maximal stability estimates.*

*Proof.* The proof is by the same iteration scheme as the corresponding result for shocks, Theorem 6.18 in [GMWZ4], except that it is simpler because the function $\psi$ defining the free transmission boundary in [GMWZ4] is absent in the present fixed boundary problem. In the partially parabolic case the iteration scheme, which is explained in [GMWZ4] section 6.1.1, must be designed carefully, because the linearized estimates give weaker control over the "hyperbolic component" $u^1$ than the "parabolic component" $u^2$. Higher $W^{s,\infty}$ norms of $u^\varepsilon - u_a^\varepsilon$ can be estimated by increasing $k$, $M$, and $s_0$.

The proof of Theorem 6.18 in [GMWZ4] used the constant multiplicity assumption (H4) to construct symmetrizers. The proof of our Proposition 1.24 yields symmetrizers under the weaker assumption (H4′) when (H4) fails in the symmetric-dissipative case. Those symmetrizers are then used exactly as in the constant multiplicity case.

□

## 1.6 Verification of the Evans condition for small-amplitude layers

For large-amplitude boundary-layers, the Evans condition may be checked numerically; see [CHNZ, HLyZ1] in the one-dimensional case, [HLyZ2] in the multi-dimensional shock case. For small-amplitude layers, it may be determined analytically in several interesting cases. In particular, it holds *always* for small-amplitude layers of symmetric–dissipative systems with Dirichlet boundary conditions, and *never* (by Proposition 2.5(ii) together with Lemma 1.23) for constant layers of systems with mixed Dirichlet–Neumann conditions for which the number $N_+$ of incoming hyperbolic characteristics on $\mathcal{U}$ exceeds the number of Dirichlet conditions imposed on $\mathcal{U}^*$, or, equivalently, the number of Neumann conditions exceeds $N_b - N_+ = N_-^2$.

Our main spectral stability result is the following Theorem established in Section 3, which implies that Evans stability of small-amplitude layers $w(z)$ is equivalent to Evans stability of the constant-layer limit $w \equiv w(\infty)$, a *linear-algebraic condition* that can in principle be computed by hand. This is in sharp contrast to the shock wave case, for which the small-amplitude limit is a complicated singular-perturbation problem [Met3, PZ, FS1, FS2].

**Definition 1.26** (Small amplitude profiles)**.** *With* $\mathcal{U}_\partial$ *as in* (1.9) *define*

$$\mathcal{U}_{\partial,\nu} = \{(u, \nu(t,x_0)) : (t,x_0,u) \in \mathcal{U}_\partial\}. \tag{1.58}$$

*For $\varepsilon > 0$ and any compact set $D \subset \mathcal{U}_{\partial,\nu}$, the set of $\varepsilon$-*amplitude profiles *associated to $D$ is the set of functions $w(z) = w(z, u, \nu)$ for which there exist $(\underline{u}, \underline{\nu}) \in D$ such that:*

*a) $A_\nu(w)\partial_z w - \partial_z(B_\nu(w)\partial_z w) = 0$ on $z \geq 0$,*

*b) $w(z, u, \nu) \to u$ as $z \to \infty$,*

*c) $\|(w, w_z^2) - (\underline{u}, 0)\|_{L^\infty(0,\infty)} \leq \varepsilon$, $|\nu - \underline{\nu}| \leq \varepsilon$.*

*When $\varepsilon$ is small we refer to such profiles as* small amplitude profiles.

**Remark 1.27.** *Observe that* small amplitude profiles *are defined without specifying any boundary condition at $z = 0$. We define the Evans function for such a $w(z, u, \nu)$ using the same formula as before* (1.48)*, where $\mathbb{E}^-(\zeta)$ and $\Gamma(\zeta)$ are now defined using linearization of* (1.26) *around $w(z, u, \nu)$.*

**Theorem 1.28.** *For any compact subset $D \subset \mathcal{U}_{\partial,\nu}$ there exists an $\varepsilon > 0$ such that the uniform Evans condition is satisfied for the set of $\varepsilon$-amplitude profiles associated to D (Definition 1.26) if and only if it is satisfied for the set of constant layers $\{w(z, u, \nu) : w = u$ for all $z, (u, \nu) \in D\}$.*

As a corollary we obtain the following result, established by energy estimates in Section 4, which establishes uniform Evans stability for small amplitude layers in a variety of situations.

**Corollary 1.29.** *(a) In the strictly parabolic case ($N = N'$) the uniform Evans condition is satisfied for sufficiently small-amplitude layers (in the sense of Definition 1.26) of symmetric–dissipative systems with pure Dirichlet boundary conditions,*

$$\text{rank}\Upsilon^3 = 0,$$

*or with pure Neumann boundary conditions when* $\text{rank}\Upsilon^3 = N = N_-^2$.

*(b)In the partially parabolic case ($N' < N$), the uniform Evans condition is satisfied for sufficiently small-amplitude layers of symmetric–dissipative systems with pure Dirichlet boundary conditions or with mixed boundary conditions when* $\text{rank}\Upsilon^3 = N' = N_-^2$ *and $\overline{A}_\nu^{11}$ is* totally outgoing *($N_+^1 = 0$).*

*(c)In the partially parabolic case when* $\text{rank}\Upsilon^3 = N' = N_-^2$ *and $\overline{A}_\nu^{11}$ is* totally incoming *($N_+^1 = N - N'$), Evans stability fails in general even for small amplitude profiles (see Example 4.3).*

*(d)The uniform Evans condition fails for sufficiently small amplitude solutions with mixed boundary conditions whenever* $\text{rank}\Upsilon^3 > N_-^2$*; see Corollary 2.5.*

Combining Theorem 1.25, Proposition 2.6, and Corollary 1.29, we obtain the following result asserting existence and stability of small-amplitude boundary-layer solutions for symmetric–dissipative systems with various types of boundary conditions. Suppose we are given a smooth, global assignment of states

$$(t, x_0, p(t, x_0)) \in \mathcal{U}_\partial \text{ for all } (t, x_0) \in [-T, T] \times \partial\Omega \tag{1.59}$$

satisfying the viscous boundary condition (1.4):

$$(\Upsilon_1(p^1(t,x_0)), \Upsilon_2(p^2(t,x_0), \Upsilon_3(p(t,x_0),0,0)) = (g_1(t,x_0), g_2(t,x_0), 0) \tag{1.60}$$

for each $(t,x_0)$. Note that the image of $p$ is compact by compactness of $\partial\Omega$.

**Theorem 1.30.** *Consider a symmetric-dissipative system* (1.26) *with boundary conditions of the type described in Corollary 1.29 (a),(b) under hypotheses (H1)-(H6), but with (H4′) in place of (H4). Given a smooth global assignment of states $p(t,x_0)$ as in* (1.59)–(1.60)*, there exists a $\mathcal{C}$ manifold satisfying Assumption* (1.12) *with $p(t,x_0) \in \mathcal{C}(t,x_0) \subset \pi\mathcal{U}_\partial$ for all $(t,x_0)$, and associated small amplitude profiles $W(z,t,x_0,q)$ satisfying the uniform Evans condition on $[-T,T]\times\partial\Omega$. The manifold $\mathcal{C}$ defines a residual hyperbolic boundary condition as in* (1.27)*.*

*Given initial data $v^0$ satisfying appropriate corner compatibility conditions for the hyperbolic problem* (1.27)*, there exists an inviscid solution $u^0$ as in Theorem 1.17, an approximate solution $u_a^\varepsilon$ as in Proposition 1.35, and an exact boundary layer solution $u^\varepsilon$ satisfying all the conclusions of Theorem 1.25 for constants $s_0$, $k$, $M$ as described there.*

## 1.7 Application to *identifying* small viscosity limits.

The exact viscous solutions in Theorems 1.25 and 1.30 are chosen to satisfy high-order corner compatibility conditions at $t=0$ that depend on the approximate solution $u_a^\varepsilon$; the construction of $u_a^\varepsilon$ depends in turn on having a $\mathcal{C}$ manifold with associated layer profiles and an inviscid solution $u^0$ to start with. In this section we show how Theorem 1.25 can sometimes be used together with Corollary 1.29 and our results on $\mathcal{C}$ manifolds in section 2 to identify small viscosity limits of solutions to viscous boundary problems like (1.26) *even when neither the $\mathcal{C}$ manifold nor the inviscid solution is given in advance.*

Consider the viscous problem (1.26) on a half-space $\Omega = \{x \in \mathbb{R}^d : x_d \geq 0\}$:

$$\begin{aligned}
&\mathcal{L}_\varepsilon(u) := A_0(u)u_t + \sum_{j=1}^d A_j(u)\partial_j u - \varepsilon \sum_{j,k=1}^d \partial_j\big(B_{jk}(u)\partial_k u\big) = 0,\\
&\Upsilon(u, \partial_T u^2, \partial_\nu u^2) = (g_1, g_2, 0) \text{ on } [-T,T]\times\partial\Omega,\\
&u = \underline{u} \in \pi\mathcal{U}_\partial \text{ in } t<0,
\end{aligned} \tag{1.61}$$

where $\underline{u}$ is a constant state and the (nonconstant) boundary data $(g^1(t,x_0), g^2(t,x_0), 0)$ is $C^\infty$, equal in $t<0$ to the constant

$$\Upsilon(\underline{u}, \partial_T\underline{u}, \partial_\nu\underline{u}) = \Upsilon(\underline{u},0,0) := (\underline{g},0), \tag{1.62}$$

and also equal to $(\underline{g},0)$ outside a compact set in $[-T,T]\times\Omega$. We suppose that (1.26) is a symmetric-dissipative system satisfying (H1)-(H6) with (H4′) in place of (H4), and that the boundary conditions are of the type described in Corollary 1.29(a)(b).

Set $\underline{\nu} = (0,1)$ and note that the constant profile $w(z) \equiv \underline{u}$ satisfies the uniform Evans condition by Corollary 1.29; so in particular, it is transversal. Now apply Proposition 2.8

to find neighborhoods $\mathcal{O} \subset \mathbb{R}^N$ of $\underline{u}$ and $\mathbb{O} \subset \mathbb{R}^{N_b - N''}$ of $\underline{g}$, smoothly varying manifolds $\mathcal{C}_{\underline{\nu},g} \subset \mathcal{O}$ for $g \in \mathbb{O}$, and transversal small-amplitude profiles

$$W_{\underline{\nu},g}(\cdot, q) : [0, \infty) \times \mathcal{C}_{\underline{\nu},g} \to \pi\mathcal{U}_* \tag{1.63}$$

such that for each $q \in \mathcal{C}_{\underline{\nu},g}$, $w = W_{\underline{\nu},g}(\cdot, q)$ satisfies

$$\begin{aligned} &A_{\underline{\nu}}(w)\partial_z w - \partial_z(B_{\underline{\nu}}(w)\partial_z w) = 0 \text{ on } z \geq 0 \\ &\Upsilon(w, 0, w_z^2)(0) = (g, 0) \\ &w(z) \to q \text{ as } z \to \infty. \end{aligned} \tag{1.64}$$

Provided $T_0$ is sufficiently small, our assumptions on the boundary data imply that

$$(g^1(t, x_0), g^2(t, x_0)) \in \mathbb{O} \text{ for all } (t, x_0) \in [-T_0, T_0] \times \partial\Omega. \tag{1.65}$$

If we now define

$$\begin{aligned} &\mathcal{C}(t, x_0) := \mathcal{C}_{\underline{\nu},g(t,x_0)} \text{ and} \\ &W(z, t, x_0, q) := W_{\underline{\nu},g(t,x_0)}(z, q) \text{ for } (t, x_0) \in [-T_0, T_0] \times \partial\Omega, \end{aligned} \tag{1.66}$$

the manifolds $\mathcal{C}(t, x_0)$ and profiles $W(z, t, x_0, q)$ satisfy the conditions of Assumption 1.12. Moreover, by Corollary 1.29 the uniform Evans condition holds on $[-T_0, T_0] \times \partial\Omega$.

Next, recalling Lemma 1.23 we apply Theorem 1.17 to construct the unique solution $u^0(t, x) \in H^{s_0}([-T_0, T_0] \times \Omega)$ to the inviscid hyperbolic problem

$$\begin{aligned} &\mathcal{L}_0(u^0) = 0 \text{ on } [-T_0, T_0] \times \Omega \\ &u^0(t, x_0) \in \mathcal{C}(t, x_0) \text{ on } [-T_0, T_0] \times \partial\Omega, \\ &u^0 = \underline{u} \text{ in } t < 0. \end{aligned} \tag{1.67}$$

on a possibly shorter time interval. From (1.62) and (1.66) we see that

$$\mathcal{C}(t, x_0) = \mathcal{C}_{\underline{\nu},\underline{g}} \text{ in } t < 0, \tag{1.68}$$

so corner compatibility conditions are satisfied in (1.67) to infinite order.

We now apply Theorem 1.25 to obtain approximate and exact solutions $u_a^\varepsilon$ and $u^\varepsilon$ to the viscous problem (1.61) satisfying the estimates (1.57) for constants $\varepsilon_0$, $M$, $k$, and $s_0$ as in that Theorem. Note that $u^\varepsilon$, $u_a^\varepsilon$, and $u^0$ are all equal to $\underline{u}$ in $t < 0$. *Finally notice that smooth solutions $u^\varepsilon$ to the initial boundary value problem* (1.61) *are uniquely determined from the start.* They must therefore equal the solutions obtained from Theorem 1.25 by the above procedure. The estimates involving $u^0$ in (1.57) now allow us to identify the unique $u^0$ solution to (1.67) as the small viscosity limit of the $u^\varepsilon$.

## 1.8 Discussion and open problems

Theorem 1.30 generalizes to the case of "real", or partially parabolic viscosity, the small-amplitude results obtained in [GG] for Laplacian second-order terms, and to multi-dimensions those obtained in [R3] for one-dimensional symmetric–dissipative systems. It includes as a physical application existence of small-amplitude boundary-layer solutions for the equations of compressible gas dynamics with specified in- or outflow velocity, temperature, and, in the inflow case, specified density or pressure, a result analogous to those obtained in [TW] for the incompressible case. It includes also the corresponding result for the compressible MHD equations with specified inflow or outflow velocity, temperature, magnetic field, and, in the inflow case, density or pressure, for parameter regimes satisfying the structural conditions described in Remark 5.6. See Sections 5.1, 5.2, and 5.3 for physical examples. For a general physical survey of boundary-layer behavior, see [S].

Together with pointwise Green function analyses [YZ, NZ] showing that uniform Evans stability is sufficient for long-time (i.e., time-asymptotic) stability of planar boundary-layer profiles, Theorem 1.28 yields also long-time stability of small-amplitude planar profiles with sharp pointwise rates of decay, sharpening previous results obtained by Rousset [R3] by energy methods.

Theorem 1.25 generalizes to the case of real viscosity the large-amplitude results obtained in [MZ1] and [GR] for strictly parabolic viscosities in the multi- and one-dimensional case, respectively, giving a sharp criterion for existence and stability of boundary-layer solutions in the small-viscosity limit. Determination of Evans stability in the large-amplitude case is an outstanding open problem. (The uniform Evans condition may fail in general for large-amplitude layers, as demonstrated in [SZ].) Numerical testing of the Evans condition for large-amplitude layers in multi-dimensions would be an interesting direction for further investigation; see [CHNZ, HLyZ1] for the one-dimensional case, [HLyZ2] for the multi-dimensional shock case.

We note that small-amplitude stability for symmetric–dissipative systems might be provable for variable-multiplicity systems under weaker structural assumptions than those of Theorem 1.30, which are tailored for large-amplitude layers, by direct energy estimates as in [GG, R3] rather than by first passing to the constant-layer limit. This approach becomes quite complicated in the multidimensional case, but would yield existence without hard-to-verify structural conditions, and would apply to some physical cases such as MHD for parameter regimes different from that described in Remark 5.6.

Finally, we discuss the meaning of the somewhat unexpected instability result of Propositions 1.13 and Corollary 1.29, parts (c) and (d). In cases where $N'' > N_-^2$, or equivalently, when the number of scalar Dirichlet conditions imposed in the viscous problem $(N' - N'' + N_+^1)$ is strictly less than the number of scalar boundary conditions for the residual hyperbolic problem $(N_+)$, this seems to indicate a failure of our basic Ansatz, which assumes that the residual hyperbolic problems should involve only *Dirichlet* boundary conditions. For example, in the extreme case $N = N'$ (strict parabolicity), $N_+ = N = N_b$ (so $N_-^2 = 0$ and all characteristics are incoming), with full Neumann boundary conditions $\partial_\nu u(0) = 0$ on $\partial\Omega$ (so $N'' = N$), work in progress indicates that solutions converge in the small-viscosity limit to solutions of the inviscid problem $\mathcal{L}_0(u^0) = 0$ with *Neumann boundary conditions* $\partial_\nu u(0) = 0$ *on* $\partial\Omega$ , and with no intervening boundary layer. In other cases where $N'' > N_-^2$, we conjecture that the correct model for limiting behavior is a residual hyperbolic problem with mixed Dirichlet–Neumann conditions of appropriate ranks. We plan to address this issue in a future work.

**Plan of the paper.** In Section 2 we investigate the existence and transversality of boundary layers, in particular in the small-amplitude limit, and construct both small and large amplitude $\mathcal{C}$-manifolds. In Section 3, we review the construction of the Evans function, and examine its high-frequency and small-amplitude limits, establishing the key reduction of Theorem 1.28 for small-amplitude profiles. In Section 4, we establish Corollary 1.29 for symmetric dissipative systems using energy estimates. In Section 4.3, we digress slightly to show maximal dissipativity of hyperbolic boundary conditions associated with small-amplitude layers of symmetric dissipative systems when full Dirichlet conditions are imposed in the viscous problem ($N'' = 0$). In Sections 5.1–5.3, we carry out explicit computations for the example systems of isentropic gas dynamics, full gas dynamics, and MHD. The construction of approximate solutions is presented in Appendix A. The tracking lemma and its connection to construction of high-frequency symmetrizers are given in Appendix B.

**Notations 1.31.** *We do not distinguish between* $u^2$ *and* $u_2$*,* $J^*$ *and* $J_*$*, etc.. Sometimes, especially when other subscripts or superscripts are involved, one choice is more convenient than the other (e.g.,* $u_z^2$*).*

# 2 Existence of $\mathcal{C}$-manifolds and layer profiles

In this section we will show that it is possible to construct smooth $\mathcal{C}$-manifolds as in Assumption 1.12, globally defined on $\partial\Omega$ with corresponding global smooth families of *small-amplitude* layer profiles. We will also show that it is possible to give a local verification of Assumption 1.12 if one starts with a given, possibly large amplitude, transversal layer profile.

## 2.1 Global $\mathcal{C}$-manifolds and families of profiles in the small-amplitude case

Here we verify Assumption 1.12 in the small-amplitude case for a variety of boundary conditions of the form (1.4). We begin by defining a family of constant layers by giving a smooth presciption of states $p(t, x_0) \in \mathbb{R}^N$ with

$$(t, x_0, p(t, x_0)) \in \mathcal{U}_\partial \text{ for all } (t, x_0) \in [-T, T] \times \partial\Omega. \tag{2.1}$$

We shall not try to show that such smooth global prescriptions are always possible under structural Assumption 1.1, but they clearly exist for the Navier-Stokes and MHD equations (see sections 5.1, 5.2, and 5.3), where one has a simple characterization of the domains of hyperbolicity and non-characteristicity in terms of physical quantities like pressure and velocity. The prescription (2.1) determines a choice of boundary data in (1.4) or (1.16), namely

$$(g_1(t,x_0), g_2(t,x_0), 0) := (\Upsilon_1(p^1(t,x_0)), \Upsilon_2(p^2(t,x_0)), 0). \tag{2.2}$$

**Remark 2.1.** *When $\Omega$ is a half-space, so that $\nu$ is constant, prescription* (2.1) *is trivially constructed, consisting of a single state $p$.*

Given $p(t,x_0)$ as in (2.1), define the compact set

$$B := \{(\nu(x_0), p(t,x_0)) : (t,x_0) \in [-T,T] \times \partial\Omega\} \subset S^{d-1} \times \pi\mathcal{U}_\partial. \tag{2.3}$$

Fix $(\underline{\nu}, \underline{p}) \in B$. The next Proposition, which is a slight modification of Proposition 5.3.5 in [Met4], characterizes all possible "small-amplitude" solutions $w(z)$ of

$$\begin{aligned} &A_\nu(w)\partial_z w - \partial_z(B_\nu(w)\partial_z w) = 0 \text{ on } z \geq 0, \\ &w(z) \to q \text{ as } z \to \infty \end{aligned} \tag{2.4}$$

for $(\nu, q)$ near $(\underline{\nu}, \underline{p})$. Define the $N' \times N'$ matrix

$$G_\nu(q) := (B_\nu^{22})^{-1}\left(A_\nu^{22} - A_\nu^{21}(A_\nu^{11})^{-1}A_\nu^{12}\right)(q) \tag{2.5}$$

and let $\mathbb{E}_\mp(G_\nu(q))$ denote the generalized eigenspace of $G_\nu(q)$ associated to eigenvalues $\mu$ with $\pm\Re\mu < 0$. Denote by $\Pi_{\nu\pm}(q)$ the projections associated to the decomposition

$$\mathbb{R}^{N'} = \mathbb{E}_+(G_\nu(q)) \oplus \mathbb{E}_-(G_\nu(q)), \tag{2.6}$$

and fix isomorphisms $\alpha(\nu, q; a)$ linear in $a \in \mathbb{E}_-(G_\nu(\underline{p}))$ and $C^\infty$ in $(\nu, q)$:

$$\alpha(\nu, q; a) : \mathbb{E}_-(G_\nu(\underline{p})) \to \mathbb{E}_-(G_\nu(q)) \tag{2.7}$$

such that $\alpha(\nu, \underline{p}; a) = a$.

**Proposition 2.2.** *There exists a neighborhood $\omega \subset S^{d-1} \times \mathbb{R}^N$ of $(\underline{\nu}, \underline{p})$ and constants $R > 0$, $r > 0$ such that for $(\nu, q) \in \omega$, all solutions $w$ of* (2.4) *satisfying*

$$\|(w, w_z^2) - (\underline{p}, 0)\|_{L^\infty[0,\infty]} \leq R, \tag{2.8}$$

*are parametrized by a $C^\infty$ function $w = \Phi(z, \nu, q, a)$ on $[0,\infty) \times \omega^*$, where $\omega^*$ is the set of $(\nu, q, a)$ with $(\nu, q) \in \omega$ and $a \in \mathbb{E}_-(G_\nu(\underline{p}))$ with $|a| \leq r$. The function $\Phi(z, \nu, q, a)$ is the unique solution of* (2.4) *satisfying the boundary condition*

$$\Pi_{\nu-} w_z^2(0) = \alpha(\nu, q; a), \tag{2.9}$$

*and $\Phi^2$ has the expansion*

$$\Phi^2(z, \nu, q, a) = q^2 + e^{zG_\nu(q)} G_\nu^{-1}(q)\alpha(\nu, q; a) + O(|a|^2) \tag{2.10}$$

*uniformly with respect to* $(z, q, \nu)$. *Moreover, there exist positive constants* $\delta$ *and* $C$ *such that for all* $z \in [0, \infty)$ *and* $(q, \nu, a) \in \omega^*$*:*

$$|\partial_z \Phi^2(z, \nu, q, a)| + |\Phi(z, \nu, q, a) - q| \leq C e^{-\delta z}. \tag{2.11}$$

*We also denote by* $\Phi(z, \nu, q, a)$ *the maximal extension of* $\Phi$ *to* $z < 0$ *as a solution of* (2.4).

*Proof.* **1.** First we claim there exists a neighborhood $\omega' \subset S^{d-1} \times \mathbb{R}^N$ of $(\underline{\nu}, \underline{p})$ and constants $R' > 0$, $r' > 0$ such that for $(\nu, q) \in \omega'$, all solutions $w$ of (2.4) satisfying

$$\|w_z^2\|_{L^1[0,\infty]} \leq R' \text{ and } \|w_z^2\|_{L^\infty[0,\infty]} \leq R', \tag{2.12}$$

are parametrized by a $C^\infty$ function $w = \Phi(z, \nu, q, a)$ on $[0, \infty) \times \omega'_*$, where $\omega'_*$ is the set of $(\nu, q, a)$ with $(\nu, q) \in \omega'$ and $a \in \mathbb{E}_-(G_\nu(\underline{p}))$ with $|a| \leq r'$. This may be established by a contraction mapping argument identical to that given in Proposition 2.2 and Appendix A (both) of [GMWZ7]. This argument corrects a minor error in [Met4], Proposition 5.3.5 and extends that Proposition to the case of partial viscosity.

**2.** For $\nu$ near $\underline{\nu}$, after shrinking $\omega'$ and $r'$ if necessary and renaming as $\omega$ and $r$, the map $(q, a) \to (\Phi, \Phi_z^2)(0, \nu, q, a)$ defined for $(\nu, q) \in \omega$, $|a| \leq r$, defines a diffeomorphism onto the local center–stable manifold of $(\underline{p}, 0)$ for (2.4) considered as a first-order system (1.18). On the other hand for $\nu$ near $\underline{\nu}$ all solutions of (2.4) for which

$$\|(w, w_z^2) - (\underline{p}, 0)\|_{L^\infty[0,\infty)} \text{ is small} \tag{2.13}$$

lie on that center-stable manifold. Thus, for $R$ small enough the assertion in the Proposition holds. □

**Remark 2.3.** *Alternatively, the result of Proposition 2.2 may be obtained directly by invariant manifold theory, working with the first-order system* (1.18)*. For, noting that any constant function is an equilibrium of the system, and recalling that equilibria lie on any center manifold, we find by a dimensional count that the center manifold of the system consists entirely of equilibria, and the center–stable manifold is foliated by the union of stable manifolds through each equilibrium (constant state). Thus, the only profiles satisfying* (2.8) *are those lying on stable manifolds of rest points* $q$*, whence we obtain both the parametrization by* $\Phi$ *and the decay estimate* (2.11) *by an application of the Stable Manifold Theorem.*

The next Proposition gives necessary and sufficient conditions on the boundary conditions $\Upsilon(w, 0, \partial_z w^2)$ in (1.16) for the *local* (with respect to $(\nu, p)$) existence of transversal profiles and corresponding $\mathcal{C}$ manifolds. The local objects will then be patched together using an argument based on local uniqueness.

For $(\nu, q, a, p)$ near $(\underline{\nu}, \underline{p}, 0, \underline{p})$, define

$$\Psi(\nu, q, a, p) := \Upsilon(\Phi(0, \nu, q, a), 0, \partial_z \Phi^2(0, \nu, q, a)) - (\Upsilon_1(p^1), \Upsilon_2(p^2), 0) \tag{2.14}$$

Observe that $\Psi(\underline{\nu}, \underline{p}, 0, \underline{p}) = 0$ and that every solution of (2.4) which also satisfies

$$\Upsilon(w, 0, w_z^2)(0) = (\Upsilon_1(p^1), \Upsilon_2(p^2), 0) \tag{2.15}$$

corresponds to a solution of $\Psi(\nu, q, a, p) = 0$ and vice versa. Using (1.18) and the expansion (2.10), we readily compute the $N_b \times (N + N')$ derivative matrix

$$\Psi_{q^1,q^2,a}(\underline{\nu}, \underline{p}, 0, \underline{p}) = \begin{pmatrix} \Upsilon_1'(\underline{p}^1) & 0 & \Upsilon_1'(\underline{p}^1)\left(-(A^{11}_{\underline{\nu}})^{-1}(\underline{p})A^{12}_{\underline{\nu}}(\underline{p})G^{-1}_{\underline{\nu}}(\underline{p})\right) \\ 0 & \Upsilon_2'(\underline{p}^2) & \Upsilon_2'(\underline{p}^2)G^{-1}_{\underline{\nu}}(\underline{p}) \\ 0 & 0 & K_{\underline{\nu}} \end{pmatrix}, \tag{2.16}$$

where, for example, the matrix entries in the third column, reading down, have sizes $N_1^+ \times N'$, $(N' - N'') \times N'$, and $N'' \times N'$ respectively.

**Proposition 2.4.** *(a) For $B$ as in* (2.3)*, let $(\underline{\nu}, \underline{p}) \in B$. The constant layer $\Phi(z, \underline{\nu}, \underline{p}, 0) = \underline{p}$ is transversal if and only if*

$$\begin{aligned} &(i)\text{ the } N_b \times N' \text{ third column of (2.16) is injective on } \mathbb{E}_-(G_{\underline{\nu}}(\underline{p})), \text{ and} \\ &(ii)\text{ if } N'' > 0, K_{\underline{\nu}} \text{ is of full rank } N'' \text{ on } \mathbb{E}_-(G_{\underline{\nu}}(\underline{p})). \end{aligned} \tag{2.17}$$

*(b) Suppose* (2.17) *holds. There is a neighborhood $\omega \subset S^{d-1} \times \pi\mathcal{U}_\partial$ of $(\underline{\nu}, \underline{p})$ and for each $(\nu, p) \in \omega$, there is a manifold $\mathcal{C}_{\nu,p}$ of dimension $N - N_+$ and a smooth map*

$$w_{\nu,p} : [0, \infty) \times \mathcal{C}_{\nu,p} \to \pi\mathcal{U}_\partial, \tag{2.18}$$

*such that for each $q \in \mathcal{C}_{\nu,p}$, $w_{\nu,p}(\cdot, q)$ satisfies* (2.4),(2.15) *and converges at an exponential rate to $q$ as $z \to \infty$. Moreover, the manifolds $\mathcal{C}_{\nu,p}$ vary smoothly with $(\nu, p) \in \omega$.*

*(c) The endstate-manifolds $\mathcal{C}_{\nu,p}$ and profiles $w_{\nu,p}(\cdot, q)$ are uniquely determined by this construction for $(q, \nu, p)$ near $(\underline{p}, \underline{\nu}, \underline{p})$. More precisely, when $(\nu, p)$ lies in charts centered at two distinct base points $(\underline{\nu}_k, \underline{p}_k) \in B$, $k = 1, 2$, the corresponding manifolds $\mathcal{C}^k_{\nu,p}$ are the same near $p \in \mathcal{C}^1_{\nu,p} \cap \mathcal{C}^2_{\nu,p}$ and for each $q \in \mathcal{C}^1_{\nu,p} \cap \mathcal{C}^2_{\nu,p}$, the profiles $w^k_{\nu,p}(\cdot, q)$ constructed in the separate charts coincide.*

*Proof.* **(a).** The first transversality condition in Definition 1.11 is equivalent to injectivity of $\Psi_a(\underline{\nu}, \underline{p}, 0, \underline{p})$ on $\mathbb{E}_-(G_{\underline{\nu}}(\underline{p}))$, while the second transversality condition there is equivalent to surjectivity of

$$\Psi_{q,a}(\underline{\nu}, \underline{p}, 0, \underline{p}) : \mathbb{R}^N \times \mathbb{E}_-(G_{\underline{\nu}}(\underline{p})) \to \mathbb{R}^{N_b} \tag{2.19}$$

(for more detail see [Met4], Prop. 5.5.3). Condition (i) in (2.17) is equivalent to the first of these conditions, and in view of Assumption 1.7 condition (ii) is equivalent to the second.

**(b).** Since $\Psi_a(\underline{\nu}, \underline{p}, 0, \underline{p})$ has rank $N^2_-$ on $\mathbb{E}_-(G_{\underline{\nu}}(\underline{p}))$ and $\Psi_{q,a}$ as in (2.19) has rank $N_b = N_+ + N^2_-$, the Implicit Function Theorem implies that there exist smooth functions $q(q_-, \nu, p)$, $a(q_-, \nu, p)$, where $q_- \in \mathbb{R}^{N-N_+}$ is a vector consisting of $N - N_+$ of the coordinates of $q$, such that the solutions of $\Psi(\nu, q, a, p) = 0$ near $(\underline{\nu}, \underline{p}, 0, \underline{p})$ are given precisely by

$$(\nu, q(q_-, \nu, p), a(q_-, \nu, p), p) \text{ for } (q_-, \nu, p) \text{ near } (\underline{p}_-, \underline{\nu}, \underline{p}). \tag{2.20}$$

For each $(\nu, p)$ the manifold $\mathcal{C}_{\nu,p}$ is defined by $q = q(q_-, \nu, p)$ and the profiles are given by

$$w_{\nu,p}(z, q) = \Phi(z, \nu, q(q_-, \nu, p), a(q_-, \nu, p)). \tag{2.21}$$

**(c).** Suppose $(\nu, p)$ lies in charts centered at two different base points $(\underline{\nu}_k, \underline{p}_k) \in B$, $k = 1, 2$. Let $\mathcal{C}^k_{\nu,p}$ and $w^k_{\nu,p}$ denote the corresponding manifolds and profiles. The properties of the functions $\Phi$ and $\Psi$ described in Proposition 2.2 and the discussion following 2.14 show that each of $\mathcal{C}^k_{\nu,p}$, $k = 1, 2$ coincides near $p$ with the set of $q$ such that there exists a $w(z)$ satisfying (2.4), (2.15), and

$$\|(w, w^2_z) - (q, 0)\|_{L^\infty(0,]} \text{ is small }. \tag{2.22}$$

This description is chart-independent so the manifolds must agree near $p$.

Suppose $q \in \mathcal{C}^1_{\nu,p} \cap \mathcal{C}^2_{\nu,p}$. Working in chart 2 and using the properties of the functions $\Phi$ and $\Psi$ just referred to, we conclude that *any* small amplitude profile satisfying (2.4), (2.15), and (2.22) must be given by $w^2_{\nu,p}(z, q)$. In particular, we must have $w^1_{\nu,p}(z, q) = w^2_{\nu,p}(z, q)$. $\square$

**Corollary 2.5.** *1. For pure Dirichlet conditions ($N'' = \operatorname{rank}\Upsilon_3 = 0$), sufficiently small-amplitude layers are transversal, and likewise for mixed Dirichlet–Neumann conditions in the extreme case* $\operatorname{rank}\Upsilon_3 = N^2_- = N'$.

*2. When the number of Neumann boundary conditions* $\operatorname{rank}\Upsilon_3$ *exceeds $N^2_-$, constant layers are non-transversal. Equivalently, a necessary condition for transversality of constant layers is that the number of (scalar) Dirichlet conditions for the parabolic problem, $(N' - N'') + N^1_+$, is greater than or equal to the number of Dirichlet conditions for the residual hyperbolic problem, $N_+$.*

*Proof.* When $N'' = 0$, the condition (2.17)(i) holds since $\Upsilon_2'(\underline{p}^2)G^{-1}_{\underline{\nu}}(\underline{p})$ is an invertible $N' \times N'$ matrix. When $N'' = N' = N^2_-$, Assumption 1.7 together with the fact that $\dim \mathbb{E}_-(G_{\underline{\nu}}(\underline{p})) = N^2_-$ imply that $K_\nu$ is invertible on $\mathbb{E}_-(G_{\underline{\nu}}(\underline{p}))$. Thus, both conditions in (2.17) hold. When $N'' > N^2_-$, it is impossible for (2.17)(ii) to hold. The final assertion follows by noting

$$\begin{aligned} &N_b = N'' + (N' - N'') + N^1_+ = N_+ + N^2_- \text{ so} \\ &N'' \le N^2_- \Leftrightarrow (N' - N'') + N^1_+ \ge N_+. \end{aligned} \tag{2.23}$$

$\square$

**Proposition 2.6.** *Given a smooth global assignment of states*

$$(t, x_0, p(t, x_0)) \in \mathcal{U}_\partial \textit{ for all } (t, x_0) \in [-T, T] \times \partial\Omega, \tag{2.24}$$

*suppose that transversality (equivalently, condition* (2.17) *of Proposition 2.4) holds at every point of*

$$B := \{(\nu(x_0), p(t, x_0)) : (t, x_0) \in [-T, T] \times \partial\Omega\} \subset S^{d-1} \times \pi\mathcal{U}_\partial. \tag{2.25}$$

*Then for boundary data*

$$(g_1(t,x_0), g_2(t,x_0), 0) := \left(\Upsilon_1(p^1(t,x_0)), \Upsilon_2(p^2(t,x_0)), 0\right), \tag{2.26}$$

*there exists a global $\mathcal{C}$ manifold and family of profiles $W$ satisfying the conditions of Assumption 1.12.*

*Proof.* Use the compactness of $B$ to cover $B$ with a finite number of open patches centered at basepoints $(\underline{\nu}, \underline{p})_k$. Carry out the construction of Proposition 2.4, part (b), in each patch and use the local uniqueness described in part (c) to define manifolds $\mathcal{C}_{\nu,p}$ and profiles $w_{\nu,p}(\cdot, q)$ for all $(\nu, p)$ in the covering and smoothly varying with $(\nu, p)$. Then the conditions of Assumption 1.12 are satisfied by taking

$$\begin{aligned}&\mathcal{C}(t,x_0) := \mathcal{C}_{\nu(x_0),p(t,x_0)}\\ &W(z,t,x_0,q) := w_{\nu(x_0),p(t,x_0)}(z,q).\end{aligned} \tag{2.27}$$

$\square$

**Remark 2.7.** ***1.*** *Part (a) of Proposition 2.4 can be used to produce many examples of transversal constant layers involving mixed Dirichlet-Neumann conditions for any triple $(N'', N^2_-, N')$ satisfying $0 \leq N'' \leq N^2_- \leq N'$. Together with Proposition 2.6, this yields examples where Assumption 1.12 is satisfied for the Navier-Stokes and MHD systems with a variety of boundary conditions.*

***2.*** *In the case $\mathrm{rank}\Upsilon_3 = N^2_- = N'$, the matrix $K_\nu$ is an invertible $N' \times N'$ matrix, so the $\Upsilon_3$ boundary condition in (1.16) is equivalent to $w^2_z(0) = 0$. With (1.18) this shows that the only solutions of the profile ODE (1.16) are constant layers. Since $N' - N'' = 0$, $\Upsilon_2$ is absent and we have*

$$\mathcal{C}(t,x_0) = \{q : \Upsilon_1(q) = g_1(t,x_0)\}. \tag{2.28}$$

*Thus, the residual hyperbolic boundary conditions in this case are the same as the Dirichlet boundary conditions for the parabolic problem, with Neumann conditions ignored. The leading term in the approximate solution (1.35) is given by $\mathcal{U}^0(t,x,\frac{z}{\varepsilon}) = u^0(t,x)$, the hyperbolic solution with no intervening boundary layer.*

## 2.2 Local $\mathcal{C}$-manifold associated to a given transversal profile

**Proposition 2.8.** *Suppose that $w$ is a given, not necessarily small amplitude, transversal layer profile satisfying*

$$\begin{aligned}&(a)\ A_{\underline{\nu}}(w)\partial_z w - \partial_z(B_{\underline{\nu}}(w)\partial_z w) = 0 \text{ on } z \geq 0\\ &(b)\ \Upsilon(w, 0, w^2_z)(0) = (\underline{g}^1, \underline{g}^2, 0) := (\underline{g}, 0)\\ &(c)\ w(z) \to p \text{ as } z \to \infty.\end{aligned} \tag{2.29}$$

*Then in a neighborhood $\mathcal{O} \subset \mathbb{R}^N$ of $p$ and for $(\nu, g)$ near $(\underline{\nu}, \underline{g})$, there is a smooth manifold $\mathcal{C}_{\nu,g} \subset \mathcal{O}$ of dimension $N - N_+$ and a smooth map*

$$W_{\nu,g} : [0, \infty) \times \mathcal{C}_{\nu,g} \to \pi\mathcal{U}^*, \tag{2.30}$$

*such that for each $q \in \mathcal{C}_{\nu,g}$, $W_{\nu,g}(\cdot, q)$ satisfies (2.29)(a),(b) with $(\nu, g, q)$ in place of $(\underline{\nu}, \underline{g}, p)$ and converges at an exponential rate to $q$ as $z \to \infty$. Moreover, these manifolds and profiles vary smoothly as $(\nu, g)$ varies near $(\underline{\nu}, \underline{g})$.*

*Proof.* **1.** For $\Phi(z, \nu, q, a)$ and $\alpha(\nu, q; a)$ as in Proposition 2.2, the given profile must satisfy for some sufficiently large $z_0$

$$w(z) = \Phi(z - z_0, \underline{\nu}, p, \underline{a}) \text{ for } \underline{a} \text{ such that } \Pi_{\underline{\nu},-} w_z^2(z_0) = \alpha(\underline{\nu}, p; \underline{a}). \tag{2.31}$$

This follows from the fact that for some $z_0$ sufficiently large, the condition (2.8) of that Proposition holds with $[0, \infty)$ replaced by $[z_0, \infty)$ (see Prop. 5.3.6 of [Met4] for details).

**2.** For $(\nu, q, a, g)$ near $(\underline{\nu}, p, \underline{a}, \underline{g})$, instead of (2.14) we now define

$$\Psi(\nu, q, a, g) := \Upsilon(\Phi(0, \nu, q, a), 0, \partial_z \Phi^2(0, \nu, q, a)) - (g^1, g^2, 0) \tag{2.32}$$

and observe that $\Psi(\underline{\nu}, p, \underline{a}, \underline{g}) = 0$. Transversality of $w$ implies that

$$\begin{gathered} \Psi_a(\underline{\nu}, p, \underline{a}, \underline{g}) : \mathbb{E}_-(G_{\underline{\nu}}(p)) \to \mathbb{R}^{N_b} \text{ and} \\ \Psi_{q,a}(\underline{\nu}, p, \underline{a}, \underline{g}) : \mathbb{R}^N \times \mathbb{E}^-(G_{\underline{\nu}}(p)) \to \mathbb{R}^{N_b} \end{gathered} \tag{2.33}$$

have ranks $N_-^2$ and $N_b = N_+ + N_-^2$ respectively. The Implicit Function theorem implies that there exist smooth, locally unique, functions $q(q_-, \nu, g)$, $a(q_-, \nu, g)$, where $q_- \in \mathbb{R}^{N - N_+}$ is a vector consisting of $N - N_+$ of the coordinates of $q$, such that the solutions of $\Psi(\nu, q, a, g) = 0$ near $(\underline{\nu}, p, \underline{a}, \underline{g})$ are given precisely by

$$(\nu, q(q_-, \nu, g), a(q_-, \nu, g), g) \text{ for } (q_-, \nu, g) \text{ near } (p_-, \underline{\nu}, \underline{g}). \tag{2.34}$$

For each $(\nu, g)$ the manifold $\mathcal{C}_{\nu,g}$ is defined by $q = q(q_-, \nu, g)$ and the profiles are given by

$$W_{\nu,g}(z, q) = \Phi(z - z_0, \nu, q(q_-, \nu, g), a(q_-, \nu, g)). \tag{2.35}$$

□

## 2.3 Global $\mathcal{C}$-manifold for a family of transversal profiles

Similarly as in (2.1) for the small-amplitude case, assume that we are given a smooth prescription of large-amplitude profiles, in the form of a $C^\infty$ function

$$w(z, t, x_0) : [0, \infty) \times [-T, T] \times \partial\Omega \to \mathbb{R}^N \tag{2.36}$$

that defines a transversal layer profile for each $(t, x_0)$:

$$
\begin{aligned}
&(a)\ A_{\nu(x_0)}(w)\partial_z w - \partial_z(B_{\nu(x_0)}\partial_z w) = 0 \\
&(b)\ \Upsilon(w, 0, w_z^2)_{|z=0} = (\Upsilon^1(w^1(0, t, x_0)), \Upsilon^2(w^2(0, t, x_0)), 0) \\
&\qquad\qquad := (g^1(t, x_0), g^2(t, x_0), 0) \\
&(c)\ w(z) \to w(\infty, t, x_0) := q(t, x_0) \text{ as } z \to \infty.
\end{aligned}
\tag{2.37}
$$

**Remark 2.9.** *When $\Omega$ is a half-space, so that $\nu$ is constant, assignment* (2.24) *exists trivially, consisting of a single profile.*

**Corollary 2.10.** *Given a smooth transversal family* (2.36)*, there is a smooth manifold $\mathcal{C}$ defined as the graph*

$$\mathcal{C} = \{(t, x_0, \mathcal{C}(t, x_0)) : (t, x_0) \in [-T, T] \times \partial\Omega\} \subset \mathcal{U}_\partial, \tag{2.38}$$

*where $\mathcal{C}(t, x_0) \subset \mathbb{R}^N$ is an $N - N_+$ dimensional manifold containing $q(t, x_0)$ and consisting of states $r$ near $q(t, x_0)$ for which there exists a transversal layer profile*

$$W(z, t, x_0, r) : [0, \infty) \times \mathcal{C} \to \pi\mathcal{U}^* \tag{2.39}$$

*satisfying* (2.37) *with $q(t, x_0)$ in (c) replaced by $r$ and*

$$\|W(z, t, x_0, r) - w(z, t, x_0), \partial_z(W^2(z, t, x_0, r) - w^2(z, t, x_0))\|_{L^\infty} \text{ small }. \tag{2.40}$$

*The profiles $W(z, t, x_0, r)$ are $C^\infty$ in all arguments and converge to their endstates at an exponential rate that can be taken uniform on compact subsets of $\mathcal{C}$.*

*Proof.* **1.** Cover the compact set $B = \{(\nu(x_0), q(t, x_0)) : (t, x_0) \in [-T, T] \times \partial\Omega\}$ by a finite number of charts centered at points $(\nu, q)_k$ for which we have defined

$$\alpha(\nu, q; a) : \mathbb{E}^-(G_{\nu_k}(q_k)) \to \mathbb{E}^-(G_\nu(q)) \tag{2.41}$$

as in (2.7). Fix $(t, x_0)$. As in the proof of Proposition 2.8, we have for some $k$

$$w(z, t, x_0) = \Phi(z - T, \nu(x_0), q(t, x_0), \underline{a}), \ \ \underline{a} \in \mathbb{E}^-(G_{\nu_k}(q_k)) \tag{2.42}$$

for some large $T$, where $\underline{a}$ is such that

$$\Pi_{\nu(x_0),-}(q(t, x_0))\ \partial_z w^2(T, t, x_0) = \alpha(\nu(x_0), q(t, x_0), \underline{a}) \in \mathbb{E}^-(G_{\nu(x_0)}(q(t, x_0))), \tag{2.43}$$

with $\Pi_{\nu,-}(q)$ the projection of $\mathbb{C}^{N'}$ onto $\mathbb{E}^-(G_\nu(q))$ along $\mathbb{E}^+(G_\nu(q))$.

**2.** Setting $W(z) = (w(z), w_z^2(z))$ we rewrite (2.37)(a) as a first-order system:

$$\partial_z W = \mathcal{H}_{\nu(x_0)}(W). \tag{2.44}$$

The stable manifold of $(q(t,x_0),0)$ for (2.44) *near* $(w(z,t,x_0),w_z^2(z,t,x_0))$ *for any fixed* $z\in[0,\infty)$, can be parametrized:

$$\begin{aligned}&\mathcal{W}^s(q(t,x_0);\nu(x_0);z)=\\ &\quad\{\begin{pmatrix}\Phi(z-T,\nu(x_0),q(t,x_0),a)\\ \Phi_z^2(z-T,\nu(x_0),q(t,x_0),a)\end{pmatrix}:a\in\mathbb{E}^-(G_{\nu_k}(q_k))\text{ near }\underline{a}\}\end{aligned}\tag{2.45}$$

Similarly, the center-stable manifold of $(q(t,x_0),0)$ near $(w(z,t,x_0),w_z^2(z,t,x_0))$ is

$$\begin{aligned}&\mathcal{W}^{cs}(q(t,x_0);\nu(x_0);z)=\\ &\quad\{\begin{pmatrix}\Phi(z-T,\nu(x_0),q,a)\\ \Phi_z^2(z-T,\nu(x_0),q,a)\end{pmatrix}:q\text{ near }q(t,x_0),a\in\mathbb{E}^-(G_{\nu_k}(q_k))\text{ near }\underline{a}\}.\end{aligned}\tag{2.46}$$

If $z=T$ we get the parts of the respective manifolds near

$$(w(T,t,x_0),w_z^2(T,t,x_0)).$$

If $z=0$ we get the parts near $(w(0,t,x_0),w_z^2(0,t,x_0))$. (Here we use that $\Phi(z,\nu,q,a)$ is a maximal extension to $z<0$). *Note that for any fixed $z$, these manifolds are uniquely determined locally near $(w(z,t,x_0),w_z^2(z,t,x_0))$.*

**3.** The set

$$\{(r,0):r\in\mathcal{C}(t,x_0)\}\tag{2.47}$$

is, near $(w(\infty,t,x_0),0)$, the set of endstates at infinity under the flow of (2.44) of the $N-N_+$ dimensional initial manifold

$$\begin{aligned}&\mathbb{C}_{initial}(t,x_0):=\\ &\quad\{U=(u^1,u^2,u^3):(\Upsilon_1(u^1),\Upsilon_2(u^2),0)=(g^1(t,x_0),g^2(t,x_0),0)\}\cap\\ &\qquad\mathcal{W}^{cs}(q(t,x_0);\nu(x_0);0)\subset\mathbb{R}^{N+N'},\end{aligned}\tag{2.48}$$

where we have evaluated (2.46) at $z=0$. As a check observe that the intersection (2.48) is transversal and has dimension

$$(N+N'-N_b)+(N+N_-^2)-(N+N')=N-N_+.\tag{2.49}$$

Considering the uniqueness of $\mathcal{W}^{cs}(q(t,x_0);\nu(x_0);0)$ near

$$(w(0,t,x_0),w_z^2(0,t,x_0)),$$

this description of $\mathcal{C}(t,x_0)$ establishes that it is uniquely determined by the local construction of Proposition 2.8 near $w(\infty,t,x_0)$. This allows the global $\mathcal{C}$ manifold of Corollary 2.10 to be constructed by patching together local $\mathcal{C}$ manifolds.

**4.** The profiles $W(z,t,x_0,r)$ in (2.39) are given by

$$W(z,t,x_0,r)=W_{\nu(x_0),g(t,x_0)}(z,r)\tag{2.50}$$

for $W_{\nu,g}$ as in (2.33) of GMWZ5. For each point $(r,0)$ in (2.47), there is a unique point $U_0=(u_0,u_0^3)\in\mathbb{C}_{initial}(t,x_0)$ that is mapped to it under the flow of (2.44). $w(z)=W(z,t,x_0,r)$ is uniquely characterized by the properties that:

a)$(w, w_z)$ satisfies (2.44)
b)$\Upsilon(w, 0, w_z^2)_{|z=0} = (g^1(t, x_0), g^2(t, x_0), 0)$
c)$(w, w_z^2) \to (r, 0)$ as $z \to +\infty$.
d) $(w, w_z^2)(0) = U_0 \in \mathbb{C}_{initial}(t, x_0)$.

This characterization shows that the locally constructed profiles

$$W(z, t, x_0, r)$$

in (2.50) patch together consistently. The local construction of Prop. 2.8 then shows that they are $C^\infty$ in all arguments.

$\square$

# 3 The Evans function and asymptotic limits

We now study the Evans function and its high-frequency and small-amplitude limits. We first recall the *conjugation lemma* of [MZ1], which implies that a first-order system $U' = \mathcal{G}(z, \zeta, p)U$ whose coefficient matrix converges exponentially to its limit $\mathcal{G}(\infty, \zeta, p)$ as $z \to +\infty$, may be converted by a smooth, exponentially trivial local change of coordinates

$$U = P(z, \zeta, p)V = (I + Q(z, \zeta, p))V \tag{3.1}$$

to its limiting constant-coefficient equation $V' = \mathcal{G}(\infty, \zeta, p)V$. Here, we have adjoined to the arguments $\zeta, z$ also dependence on model parameters $p$, assumed to be at least continuous.

Let $\mathcal{G}(z, \zeta, p)$ be as in (1.45), a frequency-dependent matrix arising from linearization around a profile $w(z)$ such that for some positive constants $C$, $\beta$, uniform with respect to model parameters $p$,

$$|w(z) - w(\infty)| \le Ce^{-\beta z}, \tag{3.2}$$

and also $p \to (w, \partial_z w_2)(\cdot, p)$ is continuous as a function from $p$ to $L^\infty(0, +\infty)$. Thus, also,

$$|\mathcal{G}(z, \cdot) - \mathcal{G}(\infty, \cdot)| \le Ce^{-\beta z} \tag{3.3}$$

and $\mathcal{G}(\cdot, \zeta, p)$ is continuous as a function from $p$ to $L^\infty(0, +\infty)$.

**Lemma 3.1** ([MZ1], Lemma 2.6)**.** *Let $\beta > 0$ be as in (3.2). For all $\underline{\zeta} \in \mathbb{R}^{d+1}$ and model parameters $p$, there are a neighborhood $\omega \times P$ of $(\underline{\zeta}, p)$, a matrix $P(z, \zeta, p) = I + Q(z, \zeta, p)$ that is $C^\infty$ on $[0, \infty) \times \omega$ with derivatives uniformly continuous in $p$, and positive constants $C$, $\alpha$ with $0 < \alpha < \beta$ such that*

*(i) $P$ and $P^{-1}$ are $C^\infty$ and bounded with bounded derivatives:*

$$|\partial_z^j \partial_\zeta^k Q| \le C_{jk} e^{-\alpha z}, \tag{3.4}$$

*(ii) $P(z, \zeta, p)$ satisfies*

$$\partial_z P = \mathcal{G}(z, \zeta, p)P - P\mathcal{G}(\infty, \zeta, p) \text{ on } z \ge 0. \tag{3.5}$$

Observe that $U$ satisfies (1.45) on $z \geq 0$ if and only if $V$ defined by $U = P(z,\zeta,p)V$ satisfies

$$\partial_z V = \mathcal{G}(\infty,\zeta,p)V + P^{-1}F, \;\; \Gamma P(0,\zeta,p)V|_{z=0} = G. \tag{3.6}$$

This implies that the decaying space $\mathbb{E}^-(\zeta,p)$ as in (1.48) is exactly the image under $P(0,\zeta,p)$ of the stable subspace of $\mathcal{G}(\infty,\zeta,p)$, denoted $\mathbb{E}^-_\infty(\zeta,p)$. Thus, by the calculation of [GMWZ6], Lemma 2.12, $\mathbb{E}^-(\zeta,p)$ has dimension $N_b = \operatorname{rank}\Gamma$ for $\gamma \geq 0$, $\zeta \neq 0$. The Evans determinant (1.48)

$$D_p(\zeta) = \det(\mathbb{E}^-(\zeta,p), \ker\Gamma(\zeta,p)), \tag{3.7}$$

now denoted with additional dependence on model parameters $p$, is then well-defined on $\gamma \geq 0$, $\zeta \neq 0$ and depends smoothly on $\zeta$ and continuously (in all $\zeta$-derivatives) on $p$. We record this as a corollary, of which we shall later make important use. For quantitative bounds estimating the modulus of continuity in $p$, see [PZ] Prop. 2.4 or Cor. C.3, [HLZ].

**Remark 3.2.** *The conjugator $P(z,\zeta,p)$ is constructed by a fixed point argument as the solution of an integral equation. The exponential decay* (3.3) *is needed to make the integral equation contractive in $L^\infty[M,+\infty)$ for $M$ sufficiently large. The continuity of $P$ with respect to $p$ is then immediate, by continuous dependence on parameters of fixed point solutions, a quite general result.*

**Corollary 3.3.** *Let $w(z,p)$ be a family of layer profiles depending continuously on parameters $p$ in the sense that $p \to (w(\cdot,p), w_z^2(\cdot,p))$ is a continuous function from $p$ to $L^\infty[0,+\infty)$, and let $\Gamma(\zeta,p)$ be as in* (3.7). *Then, the Evans function* (3.7) *depends continuously on $p$.*

*Proof.* **1.** Set $W(z,p) := (w(z,p), w_z^2(z,p))$. Continuity in $p$, by boundedness of $A_j$, $B_{jk}$, and derivatives, is inherited by the coefficient matrices $\mathcal{G}(\cdot,\zeta,p)$ appearing in the linearized eigenvalue equations from continuity of $W$. Likewise, continuity of the linearized boundary operator $\Gamma(\zeta,p)$ follows from boundedness of $\Upsilon$ and derivatives. In view of our rank conditions on $\Gamma(\zeta,p)$ and the continuity of $P(0,\zeta,p)$ for all $\zeta \in \overline{\mathbb{R}}^{d+1}$, we see from the definition of the Evans function (3.7) that it is sufficient to establish continuity of $\mathbb{E}^-_\infty(\zeta,p)$ with respect to $p$ for $\zeta \neq 0$ and continuity of $\mathbb{E}^-_\infty(\hat\zeta,\rho,p)$ with respect to $p$ at $\rho = 0$ (recall (1.51),(1.52)).

**2.** For $\zeta \neq 0$, the continuity of $\mathbb{E}^-_\infty(\zeta,p)$ follows by the fact that the limiting coefficient matrix $\mathcal{G}_\infty(\zeta,p)$ has a spectral gap, whence the stable subspace varies continuously by standard matrix perturbation theory [Kat]. Continuity $\mathbb{E}^-_\infty(\hat\zeta,\rho,p)$ at $\rho = 0$ is more difficult to show and follows from the existence of $K$-families of viscous symmetrizers as in [MZ3], Theorem 3.3.

□

### 3.1 Maximal stability estimates and high-frequency scaling

We next recall from [GMWZ6] the appropriate scaling of the Evans function for high-frequencies $|\zeta| \geq R > 0$. The maximal stability estimate for (1.45) on this frequency domain is (see [GMWZ6])

$$
\begin{aligned}
(3.8)\qquad (1+\gamma)\|u^1\|_{L^2(\mathbb{R}_+)} &+ \Lambda\|u^2\|_{L^2(\mathbb{R}_+)} + \|\partial_z u^2\|_{L^2(\mathbb{R}_+)}\\
+(1+\gamma)^{\frac12}|u^1(0)| &+ \Lambda^{\frac12}|u^2(0)| + \Lambda^{-\frac12}|\partial_z u^2(0)| \le\\
&\qquad C\big(\|f^1\|_{L^2(\mathbb{R}_+)} + \Lambda^{-1}\|f^2\|_{L^2(\mathbb{R}_+)}\big)\\
&\qquad + C\big((1+\gamma)^{\frac12}|g^1| + \Lambda^{\frac12}|g^2| + \Lambda^{-\frac12}|g^3|\big),
\end{aligned}
$$

where $C$ is an independent constant and $\Lambda$ is the natural parabolic weight

$$
(3.9)\qquad \Lambda(\zeta) = \big(\tau^2 + \gamma^2 + |\eta|^4\big)^{1/4}.
$$

Together with corresponding low-frequency estimates (see (3.16) below), this yields maximal spatio-temporal stability estimates by Parseval's identity [GMWZ4, GMWZ6].

Taking $f = 0$ in (3.8) yields the necessary condition

$$
\begin{aligned}
(3.10)\qquad &(1+\gamma)^{\frac12}|u^1| + \Lambda^{\frac12}|u^2)| + \Lambda^{-\frac12}|u^3| \le\\
&\quad C\big((1+\gamma)^{\frac12}|\Gamma_1 u^1| + \Lambda^{\frac12}|\Gamma_2 u^2)| + \Lambda^{-\frac12}|\Gamma_3(\zeta)(u^2,u^3)|\big)
\end{aligned}
$$

$\forall \zeta \in \overline{\mathbb{R}}^{d+1}_+$, $|\zeta| \ge R$, $\forall U = (u^1,u^2,u^3) \in \mathbb{E}_-(\zeta)$. This can be reformulated in terms of a *rescaled Evans function* (see [MZ1]). Introduce maps defined on $\mathbb{C}^{N+N'}$ and $\mathbb{C}^{N_b}$ respectively by

$$
\begin{aligned}
(3.11)\qquad J_\zeta(u^1,u^2,u^3) &:= \big((1+\gamma)^{\frac12}u^1, \Lambda^{\frac12}u^2, \Lambda^{-\frac12}u^3\big)\\
J_\zeta(g^1,g^2,g^3) &:= \big((1+\gamma)^{\frac12}g^1, \Lambda^{\frac12}g^2, \Lambda^{-\frac12}g^3\big).
\end{aligned}
$$

Note that $J_\zeta\Gamma(\zeta)U = \Gamma^{sc}(\zeta)J_\zeta U$ with

$$
(3.12)\qquad \Gamma^{sc}U = \big(\Gamma_1 u^1, \Gamma_2 u^2, K_d u^3 + \Lambda^{-1}K_T(\eta)u^2\big)
$$

(note: decoupled, bounded). Thus (3.10) reads

$$
(3.13)\qquad \forall U \in J_\zeta\mathbb{E}^-(\zeta): \quad |U| \le C|J_\zeta\Gamma(\zeta)J_\zeta^{-1}U| = C|\Gamma^{sc}U|.
$$

Introducing the *rescaled Evans function*

$$
(3.14)\qquad D^{sc}(\zeta) := \big|\det\big(J_\zeta\mathbb{E}^-(\zeta), J_\zeta\ker\Gamma(\zeta)\big)\big| = \big|\det\big(J_\zeta\mathbb{E}^-(\zeta), \ker\Gamma^{sc}(\zeta)\big)\big|,
$$

and using Lemma 1.21, we see that this stability condition is equivalent to the following definition.

**Definition 3.4.** *(a) Given a profile $w$, the linearized equation* (1.42) *satisfies the* uniform Evans condition for high frequencies *when there are $c > 0$ and $R > 0$ such that $|D^{sc}(\zeta)| \geq c$ for all $\zeta \in \overline{\mathbb{R}}_+^{d+1}$ with $|\zeta| \geq R$.*

*(b) The linearized equation* (1.42) *satisfies the* uniform Evans condition *when there are $c > 0$ and $R > 0$ such that*

$$|D(\zeta)| \geq c \text{ for } |\zeta| \leq R \text{ and } |D^{sc}(\zeta)| \geq c \text{ for } |\zeta| \geq R. \tag{3.15}$$

For completeness, we recall also the maximal stability estimates for low- and medium-frequencies $|\zeta| \leq R$, of

$$\begin{aligned} &\varphi \|u\|_{L^2(\mathbb{R}_+)} + \|\partial_z u^2\|_{L^2(\mathbb{R}_+)} + |u(0)| + |\partial_z u^2(0)| \leq \\ &\qquad\qquad C\big(\frac{1}{\varphi}\|f\|_{L^2(\mathbb{R}_+)} + |g|\big), \end{aligned} \tag{3.16}$$

where $\varphi = (\gamma + |\zeta|^2)^{\frac{1}{2}}$, for $\zeta \in \overline{\mathbb{R}}_+^{d+1} \backslash \{0\}$, $|\zeta| \leq R$. Taking $f = 0$ yields the necessary condition $|u(0)| + |\partial_z u^2(0)| \leq C|g|$ corresponding to the standard Evans condition (1.48).

**Remark 3.5.** *By Theorem 3.9 of [GMWZ6] the uniform Evans condition for low and medium frequencies implies the maximal estimate* (3.16)*. Taking $f = 0$ in* (3.16) *and using Remark 1.22, we see that the following are equivalent for $|\zeta| \leq R$:*

*a) the uniform Evans condition for $|\zeta| \leq R$,*

*b) the full bounded frequency estimate* (3.16)*,*

*c) the trace estimate in* (3.16) *when $f = 0$.*

*Similarly, by Theorem 7.2 of [GMWZ6] the uniform Evans condition for high frequencies implies the maximal estimate* (3.8)*. Using Remark 1.22 again, we deduce the equivalence of:*

*a)the uniform Evans condition for $|\zeta| \geq R$,*

*b) the full high frequency estimate* (3.8)*,*

*c) the trace estimate in* (3.8) *when $f = 0$.*

## 3.2 The high-frequency limit

In this section we show that the rather complicated high-frequency condition of Definition 3.4 may be reduced to a simple and natural linear algebraic condition corresponding roughly to well-posedness of the principal part (1.8) of equations (1.1), frozen at $z = 0$, under boundary conditions (1.4). Using the fact that the linearized boundary conditions are fully decoupled, we prove in Theorem 3.6 that satisfaction of the uniform Evans condition for sufficiently high frequencies is equivalent to satisfaction of the uniform Evans condition for the *decoupled system*

$$\begin{aligned} &(a)\ u_t^1 + \sum_j \overline{A}_j^{11}(w(0))\partial_j u^1 = 0, \\ &(b)\ u_t^2 - \sum_{j,k} \overline{B}_{jk}^{22}(w(0))\partial_j \partial_k u^2 = 0, \end{aligned} \tag{3.17}$$

with boundary conditions $\Gamma_1$ and $(\Gamma_2, \Gamma_3)$, respectively.

Let $\mathbb{E}^-(\zeta)$ denote as before the set of initial data at $z=0$ of decaying solutions of $\partial_z U - \mathcal{G}(z,\zeta)U = 0$. Our proof of Theorem 3.6 is based on showing that in the limit as $|\zeta| \to \infty$, the space $J_\zeta \mathbb{E}^-(\zeta)$ approaches, or "tracks", the rescaled stable subspace of an appropriate frozen coefficient problem.

Since subsystems (3.17)(a)–(b) are quasi-homogeneous, the uniform Evans condition is equivalent simply to nonvanishing of the decoupled Evans functions on the unit sphere in $\gamma \geq 0$, a linear algebraic computation. This reduction to a compact set of frequencies (Corollary 3.7) is essential in our later verifications of high frequency stability.

To state these results precisely we write the first order systems obtained from (3.17)(a), (b) by Fourier-Laplace transform as

$$\begin{aligned} &(a)\partial_z u^1 - \mathcal{G}_1(\zeta)u^1 = 0,\ \Gamma_1 u^1 = g^1 \\ &(b)\partial_z U^* - \mathcal{G}_2(\zeta)U^* = 0,\ \Gamma_* U^* = (g^2, 0), \end{aligned} \tag{3.18}$$

where $U^* = (u^2, u^2_z)$ and $\Gamma_* = (\Gamma_2, \Gamma_3)$. Let $e_{-,h}(\zeta)$ and $e_{-,p}(\zeta)$ denote the stable subspaces of the matrices $\mathcal{G}_1(\zeta)$ and $\mathcal{G}_2(\zeta)$ respectively. Setting

$$\begin{aligned} &J^1_\zeta(u^1) = (1+\gamma)^{\frac{1}{2}} u^1,\ \ \Gamma_1^{sc} u^1 = \Gamma_1 u^1 \\ &J^*_\zeta(u^2, u^3) := \left(\Lambda^{\frac{1}{2}} u^2, \Lambda^{-\frac{1}{2}} u^3\right) \\ &\Gamma^{sc}_*(\zeta)(u^2, u^3) = \left(\Gamma_2 u^2, K_d u^3 + \Lambda^{-1} K_T(\eta) u^2\right), \end{aligned} \tag{3.19}$$

we define rescaled Evans functions for (3.18)(a) and (b) by

$$\begin{aligned} &D_1^{sc}(\zeta) = \det_{\mathbb{C}^{N-N'}} \left(J^1_\zeta e_{-,h}(\zeta), \ker \Gamma_1\right) \quad (= \det\left(e_{-,h}(\zeta), \ker\Gamma_1\right) \text{ clearly}), \\ &D_2^{sc}(\zeta) := \det_{\mathbb{C}^{2N'}} \left(J^*_\zeta e_{-,p}(\zeta), \ker \Gamma^{sc}_*(\zeta)\right). \end{aligned} \tag{3.20}$$

The main result of this section is the following theorem.

**Theorem 3.6.** *Let $D^{sc}$ be the rescaled Evans function defined in* (3.14)*. Then* frozen-coefficients stability *for* (3.18)*, that is, the existence of positive constants* $c_1$,$c_2$*,* $R'$ *such that*

$$|D_1^{sc}(\zeta)| \geq c_1 \text{ and } |D_2^{sc}(\zeta)| \geq c_2 \text{ for all } |\zeta| \geq R', \tag{3.21}$$

*is equivalent to the rescaled uniform Evans condition, that is, the existence of positive constants $c$, $R$ such that*

$$|D^{sc}(\zeta)| \geq c \text{ for all } |\zeta| \geq R. \tag{3.22}$$

*Proof.* **1. Frequency zones.** There are two frequency zones to consider. For $\delta > 0$ sufficiently small we define the *elliptic zone*

$$\mathcal{E}_\delta := \{\zeta = (\tau, \gamma, \eta) : \gamma \geq \delta|\zeta|, |\eta| \geq \delta|\zeta|\} \tag{3.23}$$

and the complementary *coupling zone*

$$\mathcal{C}_\delta := \{\zeta : 0 \le \gamma \le \delta|\zeta|\} \cup \{\zeta : |\eta| \le \delta|\zeta|\}. \tag{3.24}$$

Here we show how the discussion of [GMWZ6], section 7 can be adapted to prove Theorem 3.6 in the more difficult case where $\zeta \in \mathcal{C}_\delta$. The elliptic case is proved in Appendix B, which also includes a new discussion (see Proposition B.4) of the connections between symmetrizer estimates and tracking.

**2. Symbols.** Let $\Gamma^m$ denote the space of *homogeneous symbols* of order $m$, that is, $C^\infty$ functions $h(z,\zeta)$ such that for all $\alpha \in \mathbb{N}^{d+1}$, all $k \in \mathbb{N}$, and some $\theta > 0$, there are constants $C_{\alpha,k}$ such that for $|\zeta| \ge 1$,

$$\begin{aligned} &|\partial_\zeta^\alpha h| \le C_{\alpha,0}|\zeta|^{m-|\alpha|}, \text{ if } k = 0,\\ &|\partial_z^k \partial_\zeta^\alpha h| \le C_{\alpha,k} e^{-\theta z}|\zeta|^{m-|\alpha|}, \text{ if } k > 0. \end{aligned} \tag{3.25}$$

Let $P\Gamma^m$ denote the space of *parabolic symbols* of order $m$, that is, $C^\infty$ functions $h(z,\zeta)$ satisfying similar estimates with $|\zeta|$ replaced by $\Lambda$.

**3. Conjugation to block diagonal form in $\mathcal{C}_\delta$.** Consider again the linearized problem (1.45)

$$\partial_z U = \mathcal{G}(z,\zeta)U + F, \ \ \Gamma(\zeta)U(0) = G. \tag{3.26}$$

where the components of $\mathcal{G}$ are given explicitly in (B.33). Set

$$\mathcal{P}^{22} = \begin{pmatrix} 0 & |\zeta| I \\ |\zeta|^{-1}\mathcal{G}^{32} & \mathcal{G}^{33} \end{pmatrix}. \tag{3.27}$$

In Lemma 7.3 of [GMWZ6] it is shown that there exist positive constants $c$, $R$, and $\delta$ such the distance between the spectrum of $\mathcal{G}^{11}(z,\zeta)$ and the spectrum of $\mathcal{P}^{22}(z,\zeta)$ is larger than $c|\zeta|$ for $\zeta \in \mathcal{C}_\delta$, $|\zeta| \ge R$. For such $\zeta$ Lemmas 7.5 and 7.6 of [GMWZ6] use this spectral separation to construct a change of variables $U = \mathcal{V}\hat{U}$, $F = \mathcal{V}\hat{F}$ that transforms (3.26) to

$$\partial_z \hat{U} = \hat{\mathcal{G}}(z,\zeta)\hat{U} + \hat{F}, \ \ \hat{\Gamma}(\zeta)\hat{U}(0) = G. \tag{3.28}$$

Here $\hat{\mathcal{G}} = \hat{\mathcal{G}}_p + \mathcal{G}'$ and with obvious notation

$$\hat{\mathcal{G}}_p = \begin{pmatrix} \hat{\mathcal{G}}^{11} & 0 & 0 \\ 0 & 0 & I \\ 0 & \mathcal{G}^{32} - \mathcal{V}^{31}\mathcal{G}^{12} & \mathcal{G}^{33} \end{pmatrix}, \ \ \mathcal{G}' = \begin{pmatrix} \Gamma^{-1} & \Gamma^0 & \Gamma^{-1} \\ \Gamma^{-1} & \Gamma^0 & \Gamma^{-1} \\ \Gamma^0 & \Gamma^0 & \Gamma^0 \end{pmatrix}. \tag{3.29}$$

We have $\mathcal{V} = \mathcal{V}_I(z,\zeta)\mathcal{V}_{II}(z,\zeta)$, where

$$\mathcal{V}_I = \begin{pmatrix} I & 0 & 0 \\ |\zeta|^{-1}\mathcal{V}^{21} & I & 0 \\ \mathcal{V}^{31} & 0 & I \end{pmatrix}, \ \ \begin{pmatrix} I & \mathcal{V}^{12} & |\zeta|^{-1}\mathcal{V}^{13} \\ 0 & I & 0 \\ 0 & 0 & I \end{pmatrix}, \ \ \mathcal{V}^{ij} \in \Gamma^0. \tag{3.30}$$

**4. Rescaling.** It is helpful to work with an equivalent rescaled system. Let $J := J_\zeta$ be the scaling operator introduced in (3.11) (There are really two such operators, but we use $J$ to denote both.) For any matrix $M$ of size $(N+N')\times(N+N')$ or $N_b\times(N+N')$, and for any $U\in\mathbb{C}^{N+N'}$ or $U\in\mathbb{C}^{N_b}$ define

$$M_s = JMJ^{-1}, U_s = JU. \tag{3.31}$$

In expressions like $\hat{U}_s$ or $\hat{\mathcal{G}}_s$ where the order of the "hat" and scaling operators may be unclear, the scaling operator always comes last.

With $V=\hat{U}_s$ the system (3.28) is then equivalent to

$$\partial_z V = \hat{\mathcal{G}}_s(z,\zeta)V + \mathcal{V}_s^{-1}F_s,\ \ \hat{\Gamma}_s(\zeta)V(0) = G_s. \tag{3.32}$$

Observe that $\hat{\Gamma}_s = \Gamma_s\mathcal{V}_s$ and that $\Gamma_s$ is the same as the operator $\Gamma^{sc}$ defined in (3.12). We may write $\hat{\mathcal{G}}_s = \hat{\mathcal{G}}_{ps} + \mathcal{G}_s'$ where

$$\begin{aligned}
\hat{\mathcal{G}}_{ps} &= \begin{pmatrix} \hat{\mathcal{G}}^{11} & 0 & 0 \\ 0 & 0 & \Lambda \\ 0 & \Lambda^{-1}(\mathcal{G}^{32}-\mathcal{V}^{31}\mathcal{G}^{12}) & \mathcal{G}^{33}\end{pmatrix},\\
\mathcal{G}_s' &= \begin{pmatrix} \Gamma^{-1} & (1+\gamma)^{\frac12}\Lambda^{-\frac12}\Gamma^0 & (1+\gamma)^{\frac12}\Lambda^{\frac12}\Gamma^{-1} \\ \Lambda^{\frac12}(1+\gamma)^{-\frac12}\Gamma^{-1} & \Gamma^0 & \Lambda\Gamma^{-1} \\ \Lambda^{-\frac12}(1+\gamma)^{-\frac12}\Gamma^0 & \Lambda^{-1}\Gamma^0 & \Gamma^0 \end{pmatrix}.
\end{aligned} \tag{3.33}$$

It will be important to know the exact form of $\mathcal{V}_s$. Direct computation of $J\mathcal{V}J^{-1}$ gives $\mathcal{V}_s =$

$$\begin{pmatrix} 1 & (1+\gamma)^{\frac12}\Lambda^{-\frac12}\mathcal{V}^{12} & 0 \\ 0 & I & 0 \\ \Lambda^{-\frac12}(1+\gamma)^{-\frac12}\mathcal{V}^{31} & \Lambda^{-1}\mathcal{V}^{31}\mathcal{V}^{12} & I \end{pmatrix}. \tag{3.34}$$

The only entry of this matrix that is not obviously bounded as $|\zeta|\to\infty$ is the (12)-entry. By equation (7.36) of [GMWZ6] we have

$$\mathcal{V}^{12}(z,\zeta) = O(|\eta|/|\zeta|), \tag{3.35}$$

so boundedness of the (12)-entry follows from

$$(1+\gamma)^{\frac12}|\eta|/|\zeta| = \frac{(1+\gamma)^{\frac12}}{|\zeta|^{\frac12}}\frac{|\eta|}{|\zeta|^{\frac12}} \le |\eta|^{\frac12} \le \Lambda^{\frac12}. \tag{3.36}$$

A similar computation of $\mathcal{V}_s^{-1}$ shows

$$|\mathcal{V}_s|\le C,\ \ |\mathcal{V}_s^{-1}|\le C \text{ uniformly for } z\ge 0,\ \zeta\in\mathcal{C}_\delta, \tag{3.37}$$

and furthermore

$$\mathcal{V}_s \approx \begin{pmatrix} 1 & O(1) & O(1) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{V}_s^{-1} \approx \begin{pmatrix} 1 & O(1) & O(1) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } |\zeta| \text{ large.} \tag{3.38}$$

**5. Incoming-outgoing estimates.** Consider now the principal part of (3.32):

$$\partial_z V = \hat{\mathcal{G}}_{ps}(z,\zeta)V + \mathcal{V}_s^{-1}F_s, \;\; \Gamma_s \mathcal{V}_s V(0) = G_s. \tag{3.39}$$

By Lemma 7.13 of [GMWZ6] there is a smooth change of variables $\mathcal{W} \in P\Gamma^0$ such that, if we set

$$T(z,\zeta) := \begin{pmatrix} I & 0 \\ 0 & \mathcal{W} \end{pmatrix}, \tag{3.40}$$

then

$$T^{-1}\hat{\mathcal{G}}_{ps}T = \begin{pmatrix} \hat{\mathcal{G}}^{11} & 0 & 0 \\ 0 & P_+ & 0 \\ 0 & 0 & P_- \end{pmatrix} := \mathcal{G}_{B,p}(z,\zeta), \tag{3.41}$$

with $P_\pm$ having their eigenvalues satisfying $\pm\Re\mu \geq c\Lambda$. Defining $W$ by

$$V = TW, \;\; W = (w^1, w^+, w^-), \tag{3.42}$$

we can write (3.39) equivalently as

$$\partial_z W = \mathcal{G}_B(z,\zeta)W + T^{-1}\mathcal{V}_s^{-1}F_s, \;\; \Gamma_s \mathcal{V}_s TW = G_s, \tag{3.43}$$

where now $\mathcal{G}_B = \mathcal{G}_{B,p} - T^{-1}T_z$.

Define the outgoing ($W^+$) and incoming ($W^-$) parts of $W$ by

$$W^+ = \begin{pmatrix} w^1_+ \\ w^+ \\ 0 \end{pmatrix}, \quad W^- = \begin{pmatrix} w^1_- \\ 0 \\ w^- \end{pmatrix}, \tag{3.44}$$

where $w^{1+} = w^1$, $w^{1-} = 0$ when $N^1_+ = 0$, $w^{1+} = 0$, $w^{1-} = w^1$ when $N^1_+ = N - N'$. With $\|U\| := \|U\|_{L^2(\mathbb{R}_+)}$ we define norms

$$\begin{aligned} &(a)\|W\|_s = (1+\gamma)^{\frac{1}{2}}\|w^1\| + \Lambda^{\frac{1}{2}}\|w^+\| + \Lambda^{\frac{1}{2}}\|w^-\| \\ &(b)\|F\|'_s = (1+\gamma)^{-\frac{1}{2}}\|F^1\| + \Lambda^{-\frac{1}{2}}\|F^2\| + \Lambda^{-\frac{1}{2}}\|F^3\| \\ &(c)|W(0)|_s = |W(0)|. \end{aligned} \tag{3.45}$$

Proposition 7.11 and Corollary 7.14 of [GMWZ6] imply that for large enough $R$ and $\zeta \in \mathcal{C}_\delta$ with $|\zeta| \geq R$, we have the following estimates:

$$\begin{aligned} &\|W^+\|_s + |W^+(0)| \leq C\|(\partial_z - \mathcal{G}_{B,p})W^+\|'_s \\ &\|W^-\|_s \leq C\|(\partial_z - \mathcal{G}_{B,p})W^-\|'_s + |W^-(0)|. \end{aligned} \tag{3.46}$$

Suppose now that $V$ is a solution of (3.32). Let $T$ be exactly as above and define $W$ by $V = TW$, $W = (w^1, w^+, w^-)$. Now $W$ satisfies

$$\partial_z W = \mathcal{G}_B(z,\zeta)W + T^{-1}\mathcal{G}'_s TW + T^{-1}\mathcal{V}_s^{-1}F_s, \;\; \Gamma_s \mathcal{V}_s TW = G_s, \tag{3.47}$$

From (3.46) we deduce the following estimates for $W$ by treating the extra terms $T^{-1}\mathcal{G}'_s TW$ and $T^{-1}T_z W$ as forcing terms:

$$\begin{aligned} &\|W^+\|_s + |W^+(0)| \le C\|(\partial_z - \mathcal{G}_{B,p})W^+\|'_s + \varepsilon(\zeta)\|W\|_s \\ &\|W^-\|_s \le C\|(\partial_z - \mathcal{G}_{B,p})W^-\|'_s + |W^-(0)| + \varepsilon(\zeta)\|W\|_s. \end{aligned} \tag{3.48}$$

Here we have used the explicit forms of $\mathcal{G}'_s$ (3.33) and $T$. For example, the first row of $T^{-1}\mathcal{G}'_s TW$ contributes (recall (3.45)(b))

$$(1+\gamma)^{-\frac{1}{2}}\|\Gamma^{-1}w^1\| + C\|\Lambda^{-\frac{1}{2}}\Gamma^0(w^+,w^-)\| + C\|\Lambda^{\frac{1}{2}}\Gamma^{-1}(w^+,w^-)\|, \tag{3.49}$$

which is a term of the form $\varepsilon(\zeta)\|W\|_s$.

Defining the outgoing and incoming parts of $V$ by

$$V^+ = TW^+, \;\; V^- = TW^-, \tag{3.50}$$

we deduce from (3.48) and (3.37) the following estimates for solutions to (3.32):

$$\begin{aligned} &(a)\; \|V^+\|_s + |V^+(0)| \le C\|F_s\|'_s + \varepsilon(\zeta)\|V\|_s \\ &(b)\; \|V^-\|_s \le C\|F_s\|'_s + C|V^-(0)| + \varepsilon(\zeta)\|V\|_s, \end{aligned} \tag{3.51}$$

where $\varepsilon(\zeta) \to 0$ as $|\zeta| \to \infty$.

**6. The stable subspace of $\hat{\mathcal{G}}_{ps}(0,\zeta)$.** Let $E^f_-(\zeta)$ be the stable subspace of the frozen operator $\hat{\mathcal{G}}_{ps}(0,\zeta)$. From (3.41) we see that when $N^1_+ = 0$ (resp. $N^1_+ = N - N'$)

$$\begin{aligned} &E^f_-(\zeta) = \{T(0,\zeta)\begin{pmatrix} 0 \\ 0 \\ w^- \end{pmatrix} : w^- \in \mathbb{C}^{N'}\}, \\ &(\text{resp., }, E^f_-(\zeta) = \{T(0,\zeta)\begin{pmatrix} w^{1-} \\ 0 \\ w^- \end{pmatrix} : w^{1-} \in \mathbb{C}^{N-N'}, w^- \in \mathbb{C}^{N'}\}), \end{aligned} \tag{3.52}$$

From this and (3.44) we conclude that for $V^-$ as in (3.50)

$$V^-(0) \in E^f_-(\zeta). \tag{3.53}$$

Proposition 7.10 of [GMWZ6] shows that, just like $\mathcal{G}_1(\zeta)$ in (3.18)(a), the matrix $\hat{\mathcal{G}}^{11}$ has eigenvalues with only positive, respectively negative, real part if $\overline{A}^{11}_d$ is outgoing ($N^1_+ = 0$), respectively incoming ($N^1_+ = N - N'$). Moreover, for $J_* := J^*_\zeta$ as in (3.19) and $\mathcal{G}_2$ as

(3.18)(b), the (32)-entry of $\hat{\mathcal{G}}_{ps}$ differs from that of $J_*\mathcal{G}_2(\zeta)J_*^{-1}$ by a term that is $O(1)$. These observations imply that for $|\zeta|$ large

$$E_-^f(\zeta) \text{ is close to } \begin{pmatrix} J_\zeta^1 e_{-,h}(\zeta) \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ J_\zeta^* e_{-,p}(\zeta) \end{pmatrix} \quad \text{(recall (3.20))}. \tag{3.54}$$

**7. Incoming-outgoing estimates imply tracking.** Suppose that $U(z)$ is a decaying solution of

$$\partial_z - \mathcal{G}(z,\zeta)U = 0. \tag{3.55}$$

Then $U(0) \in \mathbb{E}^-(\zeta)$, or equivalently,

$$V(0) = J\mathcal{V}^{-1}U(0) \in J\mathcal{V}^{-1}\mathbb{E}^-(0). \tag{3.56}$$

From (3.51) we easily obtain

$$\begin{aligned} &\|V^-\|_s \leq C|V^-(0)| + \varepsilon(\zeta)\|V^+\|_s \\ &\|V^+\|_s + |V^+(0)| \leq C|V^-(0)| + \varepsilon(\zeta)\|V^+\|_s, \end{aligned} \tag{3.57}$$

and thus

$$|V^+(0)| \leq \varepsilon(\zeta)|V^-(0)|. \tag{3.58}$$

Using (3.58), (3.56), and (3.52) we obtain

$$J\mathcal{V}^{-1}\mathbb{E}^-(\zeta) \approx E_-^f(\zeta) \text{ for } \zeta \in \mathcal{C}_\delta,\ |\zeta| \geq R. \tag{3.59}$$

Applying $J\mathcal{V}J^{-1}$ to (3.59) we find

$$J\mathbb{E}^-(\zeta) \approx J\mathcal{V}J^{-1}E_-^f(\zeta) = \mathcal{V}_s E_-^f(\zeta). \tag{3.60}$$

Since $\mathcal{V}_s$ has the upper triangular form (3.38) as $|\zeta| \to \infty$, we conclude from (3.60) and (3.54) that

$$J\mathbb{E}^-(\zeta) \approx \begin{pmatrix} J_\zeta^1 e_{-,h}(\zeta) \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ J_\zeta^* e_{-,p}(\zeta) \end{pmatrix} \quad \text{for } |\zeta| \geq R, \zeta \in \mathcal{C}_\delta. \tag{3.61}$$

**9. Conclusion.** Since

$$\ker \Gamma_s = \begin{pmatrix} \ker \Gamma^1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ \ker \Gamma_*^{sc} \end{pmatrix}, \tag{3.62}$$

(3.61) implies the equivalence in Theorem 3.6 for $|\zeta| \geq R$, $\zeta \in \mathcal{C}_\delta$. Together with the proof for the elliptic zone $\mathcal{E}_\delta$ in Appendix B, this concludes the proof of Theorem 3.6.

$\square$

As a corollary we obtain the following simple criterion for high-frequencies,

**Corollary 3.7.** *Under (H1)–(H6), satisfaction of the high-frequency uniform Evans condition for $|\zeta| \geq R$, $R > 0$ sufficiently large, is equivalent to nonvanishing on the positive parabolic sphere $\gamma \geq 0$, $|\lambda| + |\eta|^2 = 1$ (equivalently, the positive unit sphere $\gamma \geq 0$, $|\zeta| = 1$) of the parabolic Evans function*

$$d^2(\zeta) := \det_{\mathbb{C}^{2N'}} \Big( e_{-,p}(\zeta), \ker \Gamma_*^{sc}(\zeta) \Big). \tag{3.63}$$

*Proof.* The system (3.18)(b) has the form

$$\partial_z U^* = \mathcal{G}_2(\zeta) U^* := \begin{pmatrix} 0 & 1 \\ \mathcal{M} & \mathcal{A} \end{pmatrix} U^*, \tag{3.64}$$

where $\mathcal{M}(\zeta)$ and $\mathcal{A}(\zeta)$ are quasihomogeneous of degrees two and one respectively. The equation (3.64) can be written equivalently as

$$\begin{aligned} &\partial_z (J_\zeta^* U^*) = \Lambda \hat{\mathcal{G}}_2(\zeta) J_\zeta^* U^*, \text{ where} \\ &\hat{\mathcal{G}}_2(\zeta) := \begin{pmatrix} 0 & 1 \\ \frac{\mathcal{M}}{\Lambda^2} & \frac{\mathcal{A}}{\Lambda} \end{pmatrix}. \end{aligned} \tag{3.65}$$

This shows that $U^* \in e_{-,p}(\zeta)$ if and only if $J_\zeta^* U^*$ is in the stable subspace of $\hat{\mathcal{G}}_2(\zeta)$. With $\hat{\zeta} := \zeta/\Lambda$ and writing $E_-(M)$ for the stable subspace of any matrix $M$, we thus have

$$J_\zeta^* e_{-,p}(\zeta) = E_-(\hat{\mathcal{G}}_2(\zeta)) = E_-(\hat{\mathcal{G}}_2(\hat{\zeta})) = e_{-,p}(\hat{\zeta}). \tag{3.66}$$

Since we clearly have

$$\ker \Gamma_*^{sc}(\zeta) = \ker \Gamma_*^{sc}(\hat{\zeta}), \tag{3.67}$$

it follows from (3.66) that $|D^2(\zeta)| \geq C > 0$ for $|\zeta|$ large if and only if $d^2(\zeta)$ is nonvanishing on the parabolic unit sphere.

Since the eigenvalues of $\overline{A}_d^{11}$ were assumed all positive (totally incoming) or all negative (totally outgoing), we have either $\ker \Gamma_1 = \{0\}$, $e_{-,h}(\zeta) = \mathbb{C}^{N-N'}$ or else $\ker \Gamma_1 = \mathbb{C}^{N-N'}$, $e_{-,h}(\zeta) = \{0\}$. In either case the hyperbolic stability condition is trivially satisfied. $\square$

The following result verifies high-frequency stability for many physical cases, including the applications we will consider here.

**Proposition 3.8.** *Consider a layer profile $w(z)$ as in* (1.41) *and the linearized equations about $w(z)$ given by* (1.42) *(or* (1.45)*. In the symmetric-dissipative case (Defn. 1.3) the uniform high-frequency Evans condition is satisfied either for full Dirichlet conditions* $\text{rank}\Upsilon_3 = 0$ *or full Neumann conditions* $\text{rank}\Upsilon_3 = N'$ *on the parabolic variable $u^2$.*

*Proof.* **1.** By Corollary 3.7 and Remark 1.22, the uniform high-frequency Evans condition in the case of full Dirichlet (resp. full Neumann) boundary conditions on $u^2$ is equivalent to the estimate

$$|u^2_z(0)| \le C|g| \quad (\text{resp. } |u^2(0)| \le C|h|) \tag{3.68}$$

for decaying solutions of

$$\begin{aligned} &\lambda u^2 - B^{22}_{dd} u^2_{zz} - i\sum_{k\neq d}(B^{22}_{dk} + B^{22}_{kd})\eta_k u^2_z + \sum_{j,k\neq d}\eta_j\eta_k B^{22}_{jk} = 0 \\ &u^2(0) = g \quad (\text{resp. } u^2_z(0) = h), \end{aligned} \tag{3.69}$$

where the constant $C$ in (3.68) is independent of $(\lambda, \eta)$ in the positive parabolic unit sphere. As in (3.17) the coefficients in (3.69) are evaluated at $w(0)$.

The estimates below are similar, but not identical, to those given in section 4.1. Here we highlight the differences and refer to that section for extra detail. We now take

$$|\lambda| + |\eta|^2 = 1. \tag{3.70}$$

**2. Dirichlet conditions.** Taking the real part of the inner product of $u^2$ with (3.69), we obtain after integration by parts as in (4.7):

$$(\Re\lambda + |\eta|^2)|u_2|_2^2 + |u_2'|_2^2 \le C(|g||u_2'(0)| + |\eta||g|^2). \tag{3.71}$$

Here the last term on the right is a "Garding error" that is explained just below (4.6).

Similarly, taking the real part of the inner product of $-u_2''$ with (3.69), we obtain as in (4.8)

$$(\Re\lambda + |\eta|^2)|u_2'|_2^2 + |u_2''|_2^2 \le C\left(\left(|\lambda| + |\eta|^2\right)|g||u_2'(0)| + |\eta||u_2'(0)|^2\right). \tag{3.72}$$

The small differences between the estimates here and in section 4.1 reflect the absence of the matrices $A_k$ in (3.69).

Using the Sobolev bound

$$|u_2'(0)|^2 \le |u_2'|_2|u_2''|_2 \le C_\delta|u_2'|_2^2 + \delta|u''|_2^2 \tag{3.73}$$

we immediately deduce

$$|u_2'(0)| \le C|g| \tag{3.74}$$

from (3.71), (3.72), and (1.46).

**3. Neumann conditions.** Taking inner products as above, but now taking imaginary parts in order to estimate $|\Im\lambda||u_2|_2^2$ and $|\Im\lambda||u_2'|_2^2$, we obtain after combining estimates:

$$(|\lambda| + |\eta|^2)|u_2|_2^2 + |u_2'|_2^2 \le C(|g||u_2'(0)| + |\eta||g|^2) \tag{3.75}$$

and

$$(|\lambda| + |\eta|^2)|u_2'|_2^2 + |u_2''|_2^2 \le C\left(\left(|\lambda| + |\eta|^2\right)|g||u_2'(0)| + |\eta||u_2'(0)|^2\right). \tag{3.76}$$

Using the Sobolev inequality

$$|u_2(0)|^2 \le |u_2|_2|u_2'|_2^2 \le \delta|u_2|_2^2 + C_\delta|u_2'|_2^2. \tag{3.77}$$

and the estimates on $u_2$ and $u_2'$ coming from the first terms on the left in (3.75) and (3.76) respectively, we obtain

$$|u_2(0)| \le C|h|. \tag{3.78}$$

□

**Remark 3.9.** *Recalling (see Remark 3.5) that the Evans condition is equivalent to maximal stability estimates, we find from the reduction to* (3.17) *that for decoupled boundary conditions the assumed ranks of* $\Gamma_1$ *and* $(\Gamma_2, \Gamma_3)$ *are necessary in order to obtain maximal high-frequency stability estimates. For example, specifying density or pressure rather than velocity for outgoing isentropic gas-dynamical flow (see Section 5.1) would result in degraded stability estimates.*

Note that the above results hold for *arbitrary-amplitude layers*.

### 3.3 The small-amplitude limit

With Corollary 3.7 we are now able to verify the uniform Evans condition for high frequencies (Definition 3.4) by reducing to the consideration of a bounded set of frequencies. This puts us in position to prove Theorem 1.28.

*Proof of Theorem 1.28.* **1. Preliminaries.** It is sufficient to show that uniform Evans stability of the constant layers $w(z, u, \nu) \equiv u$, $(u, \nu) \in D$ implies uniform Evans stability for sufficiently small amplitude profiles associated to elements of $D$. (The reverse direction is trivial, zero-amplitude being included in the set of small-amplitude profiles.) By compactness of $D$, it is sufficient to establish stability of small-amplitude layers in the vicinity of the constant layer $w(z) = w(z, \underline{u}, \underline{\nu}) \equiv \underline{u}$ associated to a single element $(\underline{u}, \underline{\nu}) \in D$. Recall that $\varepsilon$-amplitude profiles $w(z, u, \nu)$ as in Definition 1.26 satisfy

$$\begin{aligned} &(a)\ A_\nu(w)\partial_z w - \partial_z(B_\nu(w)\partial_z w) = 0 \text{ on } z \ge 0 \\ &(b)\ w \to u \text{ as } z \to \infty, \\ &(c)\ \|(w, w_z^2) - (\underline{u}, 0)\|_{L^\infty} \le \varepsilon,\ |\nu - \underline{\nu}| \le \varepsilon \end{aligned} \tag{3.79}$$

for some $\varepsilon > 0$.

**2. Parameters.** From the assumed Evans stability of $w \equiv \underline{u}$, we have transversality of the constant layer by Lemma 1.23. Thus, by Proposition 2.2 specialized to the vicinity of a single point, there exists a neighborhood $\omega \subset \mathbb{R}^N \times S^{d-1}$ of $(\underline{u}, \underline{\nu})$ and constants $R > 0$, $r > 0$ such that for $(\nu, q) \in \omega$, all solutions $w$ of (3.79)(a),(b) satisfying

$$\|(w, w_z^2) - (\underline{u}, 0)\|_{L^\infty[0,\infty]} \leq R,$$

are parametrized by a $C^\infty$ function $w = \Phi(z, \nu, q, a)$ on $[0, \infty) \times \omega^*$, where $\omega^*$ is the set of parameters $(\nu, q, a)$ with $(\nu, q) \in \omega$ and $a \in \mathbb{E}_-(G_{\underline{\nu}}(\underline{p}))$ with $|a| \leq r$.

Let $D_{\underline{u},\underline{\nu}}(\zeta)$ denote the Evans function for (1.26) corresponding to linearization around the constant state $\underline{u}$, let $D_{\nu,q,a}(\zeta)$ be the Evans function arising from a profile $w = \Phi(z, \nu, q, a)$. and note that

$$D_{\underline{u},\underline{\nu}} = D_{\underline{\nu},\underline{u},0}. \tag{3.80}$$

Using similar notation for the Evans function $d^2(\zeta)$ defined in (3.63) we have:

$$\begin{aligned} &(a) d^2_k(\zeta) = \det\left(e_{-,p,k}(\zeta), \ker \Gamma^{sc}_{*,k}(\zeta)\right), \quad k = (\underline{u}, \underline{\nu}),\ (\nu, q, a) \\ &(b) D_k(\zeta) = \det\left(\mathbb{E}^-_k(\zeta), \ker \Gamma_k(\zeta)\right), \quad k = (\underline{u}, \underline{\nu}),\ (\nu, q, a). \end{aligned} \tag{3.81}$$

Observe that for $\nu$ near $\underline{\nu}$, given $\delta > 0$ there exists $0 < \varepsilon < R$ such that

$$\|(w, w_z^2) - (\underline{u}, 0)\|_{L^\infty[0,\infty]} \leq \varepsilon, \quad |\nu - \underline{\nu}| < \varepsilon \Rightarrow |\nu - \underline{\nu}| + |q - \underline{u}| + |a| < \delta. \tag{3.82}$$

This follows from the fact that by Proposition 2.2, for $\nu$ near $\underline{\nu}$, $(\Phi, \Phi_z^2)$ defines a diffeomorphism from a neighborhood of $(q, a) = (\underline{u}, 0)$ into the center-stable manifold of $(\underline{u}, 0)$ for (3.79)(a), written as a first-order system.

**2. High frequencies.** By Proposition 3.7 the uniform Evans condition for high frequencies is equivalent to the existence of $c > 0$ such that $|d^2(\zeta)| \geq c$ for $\zeta$ on the positive unit sphere $S^+ := \{\zeta : |\zeta| = 1, \gamma \geq 0\}$, a compact set. Thus, it suffices to show that in a small neighborhood of any $\underline{\zeta} \in S^+$, the subspaces $e_{-,p,k}(\zeta)$, $k = \underline{u}, (\nu, q, a)$ (resp. $\ker \Gamma^{sc}_{*,k}(\zeta)$, $k = \underline{u}, (\nu, q, a)$) are close when $|\nu - \underline{\nu}| + |q - \underline{u}| + |a|$ is small enough. Recall from (3.17)(b) and (1.47) that those spaces depend on the profile only through $w(0)$. When $|\nu - \underline{\nu}| + |q - \underline{u}| + |a|$ is small, we have $w(0) \approx \underline{u}$, so

$$d^2_{\nu,q,a}(\zeta) \approx d^2_{\underline{u},\underline{\nu}}(\zeta). \tag{3.83}$$

**3. Bounded frequencies**. By compactness it suffices to show that for $\zeta$ near some fixed $\underline{\zeta}$, the corresponding spaces appearing in (3.81)(b) are close for $|\nu - \underline{\nu}| + |q - \underline{u}| + |a|$ sufficiently small. This is true for the spaces $\ker \Gamma_k(\zeta)$, $k = (\underline{u}, \underline{\nu})$, $(\nu, q, a)$, since they depend on the profile only through $w(0)$. The treatment of $\mathbb{E}^-_k(\zeta)$ requires the conjugator $P(z, \zeta, k)$ of Lemma 3.1, where now we write $k$ instead of $p$ for parameters..

For $\zeta \neq 0$ let $\mathbb{E}^-_\infty(\zeta, k)$ be the stable subspace of $\mathcal{G}(\infty, \zeta, k)$, which depends on the profile only through $w(\infty)$, and recall from the discussion below (3.6) that

$$\mathbb{E}^-(\zeta, k) = P(0, \zeta, k)\mathbb{E}^-_\infty(\zeta, k), \quad k = (\nu, q, a),\ (\underline{u}, \underline{\nu}). \tag{3.84}$$

The dependence of $P(0,\zeta,k)$ on the profile is *not* through $w(0)$ alone but on the entire profile. However, recalling from Lemma 3.1 and (the proof of) Corollary 3.3 that $P(0,\zeta,k)$ and $\mathbb{E}_\infty^-(\zeta,k)$ depend continuously on the parameter $k$, we conclude from (3.84) and the definition of the Evans function that

$$D_{\nu,q,a}(\zeta) \approx D_{\underline{u},\underline{\nu}}(\zeta),$$

for $|\nu-\underline{\nu}|+|q-\underline{u}|+|a|$ sufficiently small. For $\zeta$ near 0, we replace $\mathbb{E}_\infty^-(\zeta,k)$ by $\mathbb{E}_\infty^-(\hat{\zeta},\rho,k)$ (recall (1.51)) in this argument. (This bounded frequency argument is the same as the proof of Corollary 3.3, but with the relevant parameters now explicitly identified.) □

# 4 Uniform Evans stability of small-amplitude layers for symmetric–dissipative systems

In this section we prove Corollary 1.29, which shows that the uniform Evans condition Definition 3.4(b) holds for small-amplitude layers for symmetric-dissipative systems under several types of boundary conditions. By Theorem 1.28 it suffices to show stability of constant layers for symmetric–dissipative systems.

## 4.1 The strictly parabolic case

For clarity, we first carry out the simpler strictly parabolic case.

*Proof of Corollary 1.29 in the case $N = N'$.*

Instability for $\text{rank}\Upsilon_3 > N_-^2$ follows again by Proposition 2.5 combined with Lemma 1.23,.

**1. Dirichlet boundary conditions.** By Theorem 1.28 it is sufficient to prove stability of the constant layer $w(z) = p$. The matrix $Y_2'(p)$ is now an invertible $N\times N$ matrix, so the boundary condition $Y_2'(p)u(0) = h$ can be written $u(0) = Y_2'^{-1}(p)h := g$. With $\lambda = i\tau+\gamma$ we consider decaying solutions $u(x_d,\lambda,\eta)$ of the Fourier-Laplace transformed problem with coefficients evaluated at $p$:

$$\begin{aligned} &(a)\ \ \lambda A_0 u + A_d u' + i\sum_{k\neq d} A_k\eta_k u \\ &- B_{dd}u'' - i\sum_{k\neq d}(B_{dk}+B_{kd})\eta_k u' + \sum_{j,k\neq d}\eta_j\eta_k B_{jk}u = 0, \\ &(b)\ \ u(0) = g, \end{aligned} \tag{4.1}$$

where the $A_j$ are symmetric, $A_0$ is positive definite, and $B_{jk}$ is dissipative:

$$\Re\sum_{jk} B_{jk}\xi_j\xi_k \geq \theta|\xi|^2,\ \theta>0,\ \text{for all } \xi\in\mathbb{R}^d. \tag{4.2}$$

Note that by the coordinate change

$$u = (A_0)^{-1/2} w \tag{4.3}$$

we can take $A_0 = I$ without loss of generality, another advantage of constant-coefficients. We do this in the remainder of section 4.

By Remark 3.5 we must establish the trace estimate

$$|u'(0)| \le C\Lambda |g|, \tag{4.4}$$

where $\Lambda \sim |1, \tau, \gamma|^{1/2} + |\eta|$. In this case it is easy to treat frequencies of all sizes, so here we do not make use of the reduction to bounded frequencies effected by Corollary 3.8.

Taking the real part of the inner product of $u$ with (4.1), we obtain

$$\begin{aligned} &\Re\lambda\langle u, u\rangle - \frac{1}{2} g \cdot A_d g - \Re g \cdot B_{dd} u'(0) + \langle u', \Re B_{dd} u'\rangle \\ &\quad - \sum_{k\neq 1} \langle \eta_k u, \Re i (B_{dk} + B_{kd}) u'\rangle + \sum_{j,k\neq d} \eta_k \eta_j \langle u, \Re B_{jk} u\rangle = 0. \end{aligned} \tag{4.5}$$

By (4.2), we obtain, extending to the whole line by $u \equiv 0$ on $x \le -\frac{1}{|\eta|}$ and $u = (x|\eta|+1)u(0)$ on $-\frac{1}{|\eta|} \le x \le 0$, taking the Fourier transform, and accounting for errors introduced by extension, the Gärding inequality

$$\begin{aligned} &\langle u', \Re B_{dd} u'\rangle + \sum_{k\neq d} \langle \eta_k u, \Re i (B_{dk} + B_{kd}) u'\rangle \\ &\quad + \sum_{j,k\neq d} \eta_k \eta_j \langle u, \Re B_{jk} u\rangle \ge \theta(|u'|_2^2 + |\eta|^2 |u|_2^2) - C|\eta||u(0)|^2, \end{aligned} \tag{4.6}$$

where $\theta > 0$. Here $C|\eta||u(0)|^2$ is the error due to extension in the Gärding inequality. It is an upper bound for the left side of (4.6), computed using the explicit formula for the extension of $u$ and inner products in $x \le 0$. Combining (4.6) with (4.5), we obtain

$$(\Re\lambda + |\eta|^2)|u|_2^2 + |u'|_2^2 \le C(|g||u'(0)| + (1+|\eta|)|g|^2). \tag{4.7}$$

Similarly, taking the real part of the inner product of $-u''$ with (4.1), we obtain after integration by parts,

$$(\Re\lambda + |\eta|^2)|u'|_2^2 + |u''|_2^2 \le C\left( \left(|\lambda| + |\eta| + |\eta|^2\right) |g||u'(0)| + (1+|\eta|)|u'(0)|^2 \right). \tag{4.8}$$

Using the Sobolev bound

$$|u'(0)|^2 \le |u'|_2 |u''|_2 \le C_\delta \Lambda |u'|_2^2 + \delta |u''|_2^2/\Lambda, \tag{4.9}$$

and the estimates on $|u'|_2^2$ and $|u''|_2^2$ coming from (4.7) and (4.8) respectively, we obtain

$$\Lambda |u'(0)|^2 \le C(\Lambda^2 |g||u'(0)| + \Lambda^3 |g|^2). \tag{4.10}$$

Since

$$\Lambda^2|g||u'(0)| \le C_\delta \Lambda^3 |g|^2 + \delta\Lambda |u'(0)|^2, \tag{4.11}$$

we find that (4.10) implies

$$\Lambda |u'(0)|^2 \le C\Lambda^3 |g|^2. \tag{4.12}$$

**2. Neumann boundary conditions.** In place of (4.1)(b) we now have the boundary condition

$$u'(0) = h, \tag{4.13}$$

and we will continue to write $u(0) = g$ in what follows. By Remark 3.5 it suffices to establish the trace estimate

$$\Lambda |u(0)| \le C|h| \tag{4.14}$$

for a constant $C$ independent of $\zeta$. To shorten the argument we now make use of the reduction to bounded frequencies $|\zeta| \le R$ provided by Proposition 3.8. In this regime the estimate (4.14) is equivalent to

$$|u(0)| \le C\Lambda^2 |h|, \tag{4.15}$$

so we proceed now to prove the latter estimate.

Again we use the Sobolev inequality

$$|u(0)|^2 \le |u|_2 |u'|_2^2 \le \delta |u|_2^2 + C_\delta |u'|_2^2. \tag{4.16}$$

Letting $\Lambda' := \gamma^{\frac{1}{2}} + |\eta|$, for $\Lambda' \ge c > 0$ we have from (4.7) and (4.8)

$$\begin{aligned} |u|_2^2 &\le \frac{C}{\Lambda'^2}|g||h| + \frac{C}{\Lambda'}|g|^2 \\ |u'|_2^2 &\le C\Lambda^2 |g||h| + \frac{C}{\Lambda'}|h|^2. \end{aligned} \tag{4.17}$$

Substituting into (4.16) and taking $\delta$ small enough, we absorb the terms involving $g$ to obtain

$$|u(0)|^2 \le C\Lambda^4 |h|^2. \tag{4.18}$$

By assumption $N_-^2 = N' = N$ and the fact that $N_+ = N_b - N_-^2 = 0$, the boundary term $-A_d g \cdot g$ in (4.5) has favorable sign. Instead of (4.7) we find

$$(\Re\lambda + |\eta|^2)|u|_2^2 + |u'|_2^2 + |u(0)|^2 \le C(|g||h| + |\eta||g|^2). \tag{4.19}$$

For $\Lambda' \le c$ with $c > 0$ small enough, we can absorb the terms involving $g$ in (4.19) to deduce (4.18). This completes the proof of (4.15) for bounded frequencies.

**3. Instability.** The fact that the Evans condition fails for $\operatorname{rank}\Upsilon_3 > N_-^2$ follows by Corollary 2.5 combined with Lemma 1.23. $\square$

**Remark 4.1.** *The argument of [GG] establishing stability of small-amplitude Dirichlet profiles in the non-constant coefficient, Laplacian-viscosity case required the weighted Poincaré estimate*

$$\int_0^{+\infty} |w'(z)||u(z)|^2\, dz \le \|zw'\|_{L^1(0,+\infty)}\|u'\|_{L^2(0,+\infty)}, \tag{4.20}$$

*established there for* $u(0) = 0$*. This estimate was also used in the one-dimensional treatments of [GS, R3]. In the multidimensional case of general* $B_{jk}$*, the approach of [GG] also requires a careful estimate of the error due to extension in the application of Garding's inequality. These technicalities disappear in the constant-coefficient limit, a major simplification. However, it may be possible to use the argument of [GG] in certain cases involving variable multiplicities not covered by our structural assumptions.*

## 4.2 The partially parabolic case

We now treat the case that $B_{jk}$ is only semidefinite.

*Proof of Corollary 1.29 ($N > N'$).* Instability for $\mathrm{rank}\Upsilon_3 > N_-^2$ follows again by Corollary 2.5 combined with Lemma 1.23.

**1. Dirichlet boundary conditions.** It is sufficient to consider the constant-coefficient equation

$$\lambda u + A_d u' + i\sum_{k\neq d} A_k\eta_k u - B_{dd}u'' - i\sum_{k\neq d}(B_{dk}+B_{kd})\eta_k u' + \sum_{j,k\neq d}\eta_j\eta_k B_{jk}u = 0, \tag{4.21}$$

where $A_j$ are symmetric, $A_0$ positive definite, and $B_{jk}$ is now block-diagonal and dissipative:

$$\Re\sum \xi_j\xi_k B^{22}_{jk} > 0 \text{ for } \xi\in\mathbb{R}^d\setminus\{0\}. \tag{4.22}$$

For later reference we record the first and second components of (4.21):

$$\begin{aligned}
&(a)\ (i\tau+\gamma)u_1 + i\sum_{k\neq d}A^{11}_k\eta_k u_1 + i\sum_{k\neq d}A^{12}_k\eta_k u_2 + A^{11}_d u_1' + A^{12}_d u_2' = 0\\
&(b)\ (i\tau+\gamma)u_2 + i\sum_{k\neq d}A^{21}_k\eta_k u_1 + i\sum_{k\neq d}A^{22}_k\eta_k u_2 + A^{21}_d u_1' + A^{22}_d u_2'\\
&\quad - B^{22}_{dd}u_2'' - i\sum_{k\neq d}(B^{22}_{dk}+B^{22}_{kd})\eta_k u_2' + \sum_{j,k\neq d}\eta_j\eta_k B^{22}_{jk}u_2 = 0.
\end{aligned} \tag{4.23}$$

By Proposition 3.8, we need only consider bounded frequencies $|\zeta|\le R$; this greatly simplifies the analysis.

*Case (i) Totally outgoing flow.* We first consider the totally outgoing case $A^{11}_d < 0$, for which the boundary conditions in (4.21) are

$$u_2(0) = g. \tag{4.24}$$

By Remark 3.5 a trace estimate that is equivalent to the (bounded-frequency) Evans hypothesis is

$$|u_1(0)| + |u_2'(0)| \le C\Lambda|g|, \tag{4.25}$$

where $\Lambda \sim |1, \tau, \gamma|^{1/2} + |\eta|$. We proceed to prove (4.25) assuming (4.22) and $A_d^{11} < 0$.

Pairing (4.21) with $u$ we obtain the usual Friedrichs estimate

$$\begin{aligned}(4.26)\qquad &\gamma|u|^2 + |\eta|^2|u_2|^2 + |u_2'|^2 + |u_1(0)|^2 \le \\ &\quad C(|u_2'(0)||u_2(0)| + |u_2(0)|^2 + |\eta||u_2(0)|^2),\end{aligned}$$

where $|\cdot|$ denotes an $L^2(x)$ norm for interior terms and a $\mathbb{C}^n$ norm for boundary terms, and we have dropped the subscript "2" on interior norms. Here the last term on the right represents the Gärding error (from extension) as well as a boundary term from integration by parts. The $|u_1(0)|^2$ on the left is there because of the favorable sign of $A_d^{11}$. From (4.26) we obtain

$$|u_1(0)|^2 \le C\Lambda|g|^2 + \delta|u_2'(0)|^2. \tag{4.27}$$

Similarly, differentiating (4.21) and pairing with $u'$ we obtain

$$\begin{aligned}(4.28)\qquad &\gamma|u'|^2 + |\eta|^2|u_2'|^2 + |u_2''|^2 + |u_1'(0)|^2 \le \\ &\quad C\left(|u_2''(0)||u_2'(0)| + (1+|\eta|)|u_2'(0)|^2\right),\end{aligned}$$

where again the terms on the right represent either Gärding error or boundary terms from integration by parts.

We now examine $|u_2'(0)|$. First we have

$$|u_2'(0)|^2 \le |u_2'||u_2''| \le C_\delta\Lambda|u_2'|^2 + \frac{\delta}{\Lambda}|u_2''|^2. \tag{4.29}$$

From (4.26) we find easily

$$C_\delta\Lambda|u_2'|^2 \le C\Lambda^2|g|^2 + \delta|u_2'(0)|^2. \tag{4.30}$$

We claim that the last term in (4.29) satisfies

$$\frac{\delta}{\Lambda}|u_2''|^2 \le C\Lambda^2|g|^2 + \delta|u_2'(0)|^2 + \delta|u_1(0)|^2. \tag{4.31}$$

With (4.27) this will complete the proof of (4.25).

To analyze the last term in (4.29) we first use (4.23)(b) to estimate $|u_2''(0)|$:

$$|u_2''(0)| \le C\left(|\lambda||u_2(0)| + |\eta||u(0)| + |u_1'(0)| + |u_2'(0)| + |\eta|^2|u_2(0)| + |\eta||u_2'(0)|\right), \tag{4.32}$$

and then substitute in (4.28) to get

$$\begin{aligned}\gamma|u'|^2 + |\eta|^2|u_2'|^2 + |u_2''|^2 + |u_1'(0)|^2 \le \\ C\big((|\lambda||u_2(0)| + |\eta||u(0)| + |u_1'(0)| + |u_2'(0)| \\ + |\eta|^2|u_2(0)| + |\eta||u_2'(0)|)|u_2'(0)| + (1+|\eta|)|u_2'(0)|^2\big) \le \\ C\big(|\lambda||u_2(0)| + |\eta||u(0)| + |\eta|^2|u_2(0)| + (1+|\eta|)|u_2'(0)|\big)|u_2'(0)|.\end{aligned} \tag{4.33}$$

Here we have used the $|u_1'(0)|^2$ on the left to absorb a term, and then enlarged $C$.

From (4.33) we have

$$\frac{\delta}{\Lambda}|u_2''|^2 \le \frac{\delta}{\Lambda}\left((|\lambda||u_2(0)| + |\eta||u(0)| + |\eta|^2|u_2(0)|)|u_2'(0)|\right) + \frac{\delta}{\Lambda}(1+|\eta|)|u_2'(0)|^2, \tag{4.34}$$

and the second term on the right can be absorbed in (4.29). We have

$$\frac{\delta}{\Lambda}|\lambda||u_2(0)||u_2'(0)| \le \delta\Lambda|u_2(0)||u_2'(0)| \le \delta\Lambda^2|g|^2 + \delta|u_2'(0)|^2. \tag{4.35}$$

The other terms in the estimate of $\frac{\delta}{\Lambda}|u_2''|^2$ are similar or easier to handle, so this concludes the proof of (4.25).

*Case (ii) Totally incoming flow.* It remains to treat the totally incoming case $A_d^{11} > 0$, with full Dirichlet boundary conditions $u(0) = (u_1(0), u_2(0)) = g$. A trace estimate that is equivalent to the (bounded-frequency) Evans hypothesis is

$$|u_2'(0)| \le C\Lambda^{\frac{3}{2}}|g|, \tag{4.36}$$

where $\Lambda \sim |1, \tau, \gamma|^{1/2} + |\eta|$.

Making the same energy estimates as in the totally outgoing case, we find that the only differences are that (i) there now appears a term $C|u_1(0)|^2$ in the righthand side of (4.26), and (ii) there now appears a term $C|u_1'(0)|^2$ in the righthand side of (4.33). Difference (i) is harmless, since $C|u_1(0)|^2 \le C|g|^2$ is of the same order as $C|u_2(0)|^2$ terms already appearing on the righthand side of (4.26).

Difference (ii) can be handled by estimating $|u_1'(0)|$ using (4.23)(a) as

$$|u_1'(0)| \le |(A_1^{11})^{-1}|\big(C|\zeta||u(0)| + C|u_2'(0)|\big) \le C((|\lambda| + |\eta|)|g| + |u_2'(0)|)$$

to see that $C|u_1'(0)|^2$ contributes terms of order $C((|\lambda| + |\eta|)^2|g|^2 + |u_2'(0)|^2)$. Following the previous argument and using $(|\lambda| + |\eta|)^2 \le \Lambda^4$, we obtain (4.36) as claimed.

**2. Neumann boundary conditions.** We now assume $N_-^2 = N' = \operatorname{rank}\Upsilon_3$. Consider first the *totally outgoing* case, so $A_d^{11} < 0$ and $N_+ = N_+^1 = 0$. The boundary conditions for (4.21) are now

$$u_2'(0) = h \tag{4.37}$$

and for bounded frequencies it suffices to show

$$|u_1(0)| + |u_2(0)| \le C\Lambda^2|h|. \tag{4.38}$$

Writing $u_2(0) = g$ and $\Lambda' = \gamma^{\frac{1}{2}} + |\eta|$, since $N_+ = 0$ we have now in place of (4.26)

$$\gamma|u|^2 + |\eta|^2|u_2|^2 + |u_2'|^2 + |u(0)|^2 \le C(|u_2'(0)||u_2(0)| + |\eta||u_2(0)|^2). \tag{4.39}$$

As before for $\Lambda' \le c$ with $c > 0$ small enough, we easily absorb the terms involving $g$ in (4.39) to obtain (4.38).

To estimate $u_2(0)$ we use

$$|u_2(0)|^2 \le \delta|u_2|^2 + C_\delta|u_2'|^2. \tag{4.40}$$

For $\Lambda' \ge c > 0$ we have from (4.39) and (4.33)

$$\begin{aligned} |u_2|^2 &\le \frac{C}{\Lambda'^2}|h||g| + \frac{C}{\Lambda'}|g|^2 \\ |u_2'|^2 &\le \frac{C}{\Lambda'^2}|h|\left(|\lambda||g| + |\eta||u(0)| + |\eta|^2|g| + (1+|\eta|)|h|\right) \\ &\le C\Lambda^2|g||h| + \frac{C}{\Lambda'}|u(0)||h| + C|g||h| + \frac{C}{\Lambda'}|h|^2 \end{aligned} \tag{4.41}$$

From (4.40), (4.41) we obtain, after absorbing some terms from the right

$$|u_2(0)|^2 \le C\Lambda^4|h|^2 + C|u_1(0)||h|. \tag{4.42}$$

Substituting the estimate on $|u_1(0)|$ from (4.39) into (4.42), we easily obtain (4.38) after adding the estimates for $u_1(0)$ and $u_2(0)$ and absorbing terms.

Consider finally the *totally incoming case* where $A_d^{11} > 0$. We have rank$\Upsilon_3 = N_-^2 = N'$, so $N_+ = N_+^1 = N - N'$ and the boundary conditions for (4.21) are

$$u_1(0) = g_1, u_2'(0) = h. \tag{4.43}$$

Evans stability can fail in this case, even for small amplitude profiles. See Example 4.3.

□

**Remarks 4.2.** *1.) We note that the argument in the partially parabolic case is even simpler than the treatment of the one-dimensional case in [R3], thanks mainly to the reduction to finite frequencies and to trace rather than interior estimates. In particular, we do not require the "Kawashima-type" estimate used in [R3] to obtain an interior estimate on $|u_1'|$. Nor do we require a weighted Poincare estimate of the type* (4.20) *used in [R3].*

*2.) In the small-amplitude analysis we expect that one may drop the constant-sign assumption on $A_d^{11}$, substituting as in [R3] the assumption that that $\Gamma_1$ be maximally dissipative with respect to $A_d^{11}$, i.e., that $A_d^{11}$ be negative definite on* $\ker\Gamma_1$. *For* bounded frequencies *the above arguments go through in this case essentially unchanged. For* high frequencies *the* *exponential weights of [GMWZ6, Z3] are not necessary in the small-amplitude case. It may be possible to modify the high-frequency argument of [GMWZ6] to work without the constant-sign assumption.*

*3.) In the case of Dirichlet boundary conditions (*$\operatorname{rank}\Upsilon_3 = 0$*), the arguments of this section may be simplified still further by making use of Proposition 4.4 below. This Proposition together with Lemma 1.23 yields uniform Evans stability for small frequencies* $|\zeta| \le r$*,* $r > 0$ *sufficiently small. Using also the reduction to bounded frequencies (Cor. 3.8), it follows that in order to show uniform Evans stability we must only show nonvanishing of the Evans function* $D(\zeta)$*; that is, nonexistence of nontrivial solutions of the eigenvalue equation* (4.21) *with homogeneous forcing and boundary data,* $f = g = 0$*. Under these conditions the required estimates become almost trivial.*

**Example 4.3** (A counterexample)**.** Finally, we show that stability may fail in general for Neumann boundary conditions with $N_-^2 = N' = \operatorname{rank}\Upsilon_3$, in the *totally incoming* case $A_d^{11} > 0$. Consider the linear constant-coefficient system

$$u_t + A_1 u_{x_1} + A_2 u_{x_2} = \begin{pmatrix} 0 \\ \Delta_x u_2 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & a \\ a & b \end{pmatrix}$$

on $x_2 > 0$ with boundary conditions

$$u_1|_{x_2=0} = 0, \quad \partial_{x_2} u_2|_{x_2=0} = 0,$$

under the assumptions

$$b > 0, \quad b - a^2 < 0.$$

Then, $A_2^{11} > 0$, and $N_-^2 = 1 = \operatorname{rank}\Upsilon_3$. Seeking layer profiles for this linear system, we have immediately, since these satisfy a first-order ODE in $u_2$ with initial value at $x_2 = 0$ by the Neumann condition an equilibrium value, that these are exactly the *constant solutions*, from which we deduce immediately that the residual hyperbolic boundary condition is exactly $u_1 = 0$.

Applying now Lemma 1.23, we find that a necessary condition for uniform low-frequency stability is satisfaction of the uniform Lopatinski condition for

$$u_t + A_1 u_{x_1} + A_2 u_{x_2} = 0$$

with boundary condition $u_1|_{x_2=0} = 0$, or

$$r_1 \neq 0 \quad \text{for} \quad r \in \mathbb{E}_-\Big( - (A_2)^{-1}(\gamma + i\tau + i\eta A_1)\Big) \tag{4.44}$$

for each $\tau$, $\eta \in \mathbb{R}$, $\gamma \in \mathbb{R}^+$, where $\mathbb{E}_-$ as usual denotes the limit of the stable subspace as $\gamma \to 0^+$. But, (4.44) is clearly violated for

$$r = (0, 1)^T,\ \eta = 1,\ \gamma = 0,\ \tau = b/a,$$

in which case $-(A_2)^{-1}(\gamma + i\tau + i\eta A_1)r = \mu r$ for $\mu = i/a$. Continuing eigenvector and eigenvalue $r = r(\gamma)$, $\mu = \mu(\gamma)$ while varying $\gamma$ in the positive direction, we obtain by standard matrix perturbation theory, noting that $r^T A_2$ by symmetry is an associated left eigenvector, that

$$(d\mu/d\gamma)|_{\gamma=0} = \frac{-r^T A_2 (A_2)^{-1}\partial_\gamma(\gamma + i\tau + i\eta A_1)r}{r^T A_2 r} = -A_2^{22} < 0,$$

verifying that $r \in \mathbb{E}_-$.

Alternatively, we may recall from [MZ2] that $2\times 2$ two-dimensional hyperbolic constant-coefficient systems with both outgoing and incoming characteristics and for which $A_1$ and $A_2$ do not commute satisfy the uniform Lopatinski condition if and only if they are maximally dissipative, i.e., $A_2 < 0$ on the kernel $\mathrm{span}\{(0,1)\}$ of the residual boundary condition, or $A_2^{22} < 0$, to make the same conclusion without calculation.

Note that this example does not yield one-dimensional low-frequency instability, and in fact is one-dimensionally stable. For, taking the real part of the inner product of $u$ against the associated eigenvalue equation $\lambda u + A_2 u' = u_2''$, we obtain $\Re\lambda |u|_2^2 + |u_2'|_2^2 = 0$, yielding $u_2 \equiv 0$. Substituting into the $u_1$ equation, we find by direct computation that $u_1 = e^{-\lambda z} u_1(0) = 0$. This gives one-dimensional Evans stability for bounded frequencies; the result for high frequencies follows by Proposition 3.8.

### 4.3 Maximal dissipativity of residual hyperbolic boundary conditions

Before presenting calculations for example systems, we digress slightly to complete the picture of qualitative behavior. Transversality and uniform Lopatinski condition follow for small-amplitude profiles of symmetric dissipative systems by Evans stability combined with Lemma 1.23, under mild structural conditions on multiplicity of characteristics. Here, we note that they may alternatively be verified directly, and without any assumptions on multiplicity of characteristics. Proposition 4.4 yields the additional information that residual boundary conditions for small-amplitude layers of symmetric dissipative systems satisfy not only the uniform Lopatinski condition, but also the stronger condition of maximal dissipativity: $SA_d < 0$ on the kernel of the linearized hyperbolic boundary conditions $\Gamma_{res}$, where $S > 0$ is the symmetric positive definite matrix symmetrizing $A_j$.

This result was established first in [BRa] for the case of symmetric, strictly parabolic $B_{jk}$. It was established in Lemma 4.3.1, [Met4] for the strictly parabolic (not necessarily symmetric) case; see also Lemma 7 [BSZ]. The argument is based on dissipative integral estimates of a similar flavor to those used to establish uniform Evans stability in Section 4.

**Proposition 4.4.** *Consider the class of symmetrizable dissipative systems (Definition 1.3) and the class of decoupled boundary conditions* (1.4) *that are full Dirichlet in the parabolic variable $u_2$ ($N'' = 0$) and maximally dissipative in the hyperbolic variable $u_1$: that is, $(SA_d)^{11}$ is negative definite on* $\ker \Gamma_1$. *Then sufficiently small-amplitude noncharacteristic boundary layers are transverse, and the associated residual boundary conditions are maximally dissipative.*

*Proof.* **1.** By continuity both assertions reduce to the corresponding assertions for the limiting constant layer, say $w(z) = p = (p^1, p^2)$. Transversality was already proved in Corollary 2.5, which applies also for the case of variable multiplicities. As in section 4 (see (4.3)) we may without loss of generality take $S = A_0 = I$ and $A_\nu = A_d$ (or $\nu = (0, \dots, 0, 1)$). We must show that $A_d < 0$ on the kernel of the residual boundary condition $\Gamma_{res}$ for the linearized hyperbolic problem at $p$.

Let $C_{\nu,p}$ denote the manifold of states $q$ near $p$ such that there exists a profile $W(z, q)$ satisfying (1.16) with

$$(g_1, g_2) := (\Upsilon_1(p^1), \Upsilon_2(p^2)) \tag{4.45}$$

and $W(z, q) \to q$ as $z \to \infty$. Let $\dot{\mathcal{C}}_{\nu,p}$ be the space of $\dot{q}$ such that there is a solution $\dot{w}(z, \dot{q})$ of the linearized profile problem (1.19), (1.20) with $\dot{w}(z, \dot{q}) \to \dot{q}$ as $z \to \infty$. The entries of $\mathcal{G}_+(\nu)$ in (1.21) are now evaluated at $w(z) = p$. It is not hard to check (see [Met4], Prop. 5.5.5) that

$$T_p\mathcal{C}_{\nu,p} = \dot{\mathcal{C}}_{\nu,p} := \ker \Gamma_{res}. \tag{4.46}$$

The linearized hyperbolic problem at $p$ is

$$\begin{aligned} &v_t + \sum_{j=1}^{d} A_j(p)\partial_j v = f \\ &v|_{x_d=0} \in T_p\mathcal{C}_{\nu,p}. \end{aligned} \tag{4.47}$$

Therefore, we must show

$$A_d < 0 \text{ on } \dot{\mathcal{C}}_{\nu,p}. \tag{4.48}$$

**2.** Set $\Gamma_1 := \Upsilon_1'(p^1)$ and define

$$\mathcal{N} = \{(n, 0) \in \mathbb{R}^N : n \in \ker \Gamma_1\}. \tag{4.49}$$

In Lemma 4.6 below we show

$$\dot{\mathcal{C}}_{\nu,p} = \mathcal{N} \oplus \mathbb{E}_-(A_d^{-1}B_{dd}), \tag{4.50}$$

where $\mathbb{E}_-(M)$ denotes the stable subspace of $M$. Since $\mathcal{N} \subset \ker B_{dd}$ and $A_d < 0$ on $\mathcal{N}$ (recall $A_d^{11} < 0$ on $\ker \Gamma_1$), the result thus follows by Lemma 4.5 below. □

**Lemma 4.5** ([Z1])**.** *Let $A$ and $B$ be $N \times N$ matrices, where $A$ is symmetric and invertible, and $B$ is positive semidefinite, $\Re(B) \geq 0$, satisfying in addition the block structure condition* $\ker B = \ker \Re(B)$*. Given any subspace $\mathcal{N}$ on which $A$ is negative definite, then $A$ is negative definite also on the subspace $\mathbb{E}_-(A^{-1}B) \oplus (\mathcal{N} \cap \ker B)$, where $\mathbb{E}_-(M)$ denotes the stable subspace of $M$.*

*Proof.* Suppose that $x_0 \neq 0$ lies in the subspace $\mathbb{E}_-(A^{-1}B) \oplus (\mathcal{N} \cap \ker B)$, i.e.,

$$x_0 = x_1 + x_2$$

where $x_1 \in \mathbb{E}_-(A^{-1}B)$, $x_2 \in (\mathcal{N} \cap \ker B)$. Define $x(t)$ by the ordinary differential equation $x' = A^{-1}Bx, \quad x(0) = x_0$. Then $x(t) \to x_2$ as $t \to +\infty$ and thus

$$\lim_{t \to +\infty} \langle x(t), Ax(t) \rangle = \langle x_2, Ax_2 \rangle \leq 0,$$

with equality only if $x_2 = 0$. On the other hand,

$$\langle x, Ax \rangle' = 2\Re \langle A^{-1}Bx, Ax \rangle = 2\Re \langle Bx, x \rangle \geq 0,$$

yielding $\langle x_0, Ax_0 \rangle \leq \langle x_2, Ax_2 \rangle \leq 0$. Thus, $\langle x_0, Ax_0 \rangle < 0$ unless $x_2 = 0$ and $\langle \Re(B)x, x \rangle \equiv 0$, in particular, $\langle \Re(B)x_1, x_1 \rangle$, so that $x_1 \in \ker \Re(B) = \ker(B) \subset \ker A^{-1}B$. But, this is impossible, since $x_1 \in \mathbb{E}_-(A^{-1}B)$. □

It just remains to prove:

**Lemma 4.6.** *Let $\dot{\mathcal{C}}_{\nu,p}$ be as in* (4.46)*, the space of $q$ such that there is a solution $\dot{w}(z,q)$ of the linearized profile problem* (1.19), (1.20) *with $\dot{w}(z,q) \to q$ as $z \to \infty$. Then*

$$\dot{\mathcal{C}}_{\nu,p} = \mathcal{N} \oplus \mathbb{E}_-(A_d^{-1}B_{dd}), \tag{4.51}$$

*for $\mathcal{N}$ as in* (4.49)*.*

*Proof.* **1.** With $w(z) = p$ the constant layer, define as in (2.5) the $N' \times N'$ matrix

$$G_d(p) := (B_{dd}^{22})^{-1} \left( A_d^{22} - A_d^{21}(A_d^{11})^{-1}A_d^{12} \right)(p). \tag{4.52}$$

A short computation shows

$$\mathbb{E}_-(A_d^{-1}B_{dd}(p)) = \{ \begin{pmatrix} -(A_d^{11})^{-1}A_d^{12}r^2 \\ r^2 \end{pmatrix} : r^2 \in \mathbb{E}_-(G_d(p)) \}. \tag{4.53}$$

**2.** Consider the linearized profile equation (1.19) at $p$ with $\dot{W} = (\dot{w}_1, \dot{w}_2, \dot{w}_3)$. For any $(q^1, q^2) \in \mathbb{R}^N$, this equation is easily integrated to yield a solution with $(\dot{w}_1(z), \dot{w}_2(z)) \to (q^1, q^2)$ as $z \to \infty$:

$$\begin{aligned} \dot{w}_1(z) &= -(A_d^{11})^{-1}A_d^{12}e^{zG_d(p)}(G_d(p))^{-1}r^2 + q^1 \\ \dot{w}_2(z) &= e^{zG_d(p)}(G_d(p))^{-1}r^2 + q^2 \\ \dot{w}_3(z) &= e^{zG_d(p)}r^2, \text{ where } r^2 \in \mathbb{E}_-(G_d(p)). \end{aligned} \tag{4.54}$$

Setting $\Upsilon_1'(p^1)\dot{w}_1(0) = 0$ and $\Upsilon_2'(p^2)\dot{w}_2(0) = 0$ we find

$$\begin{aligned} \Upsilon_1'(p^1)q^1 &= \Upsilon_1'(p^1)(A_d^{11})^{-1}A_d^{12}(G_d(p))^{-1}r^2 \\ q^2 &= -(G_d(p))^{-1}r^2. \end{aligned} \tag{4.55}$$

This gives

$$\dot{\mathcal{C}}_{\nu,p} = \{(q^1, q^2) : \Upsilon_1'(p^1)\left(q^1 + (A_d^{11})^{-1}A_d^{12}q^2\right) = 0, q^2 \in \mathbb{E}_-(G_d(p))\}. \tag{4.56}$$

Together with (4.53), this implies the result. □

**Remark 4.7.** *A theorem of Rauch [Ra] asserts the converse result that any maximally dissipative boundary condition may be realized as the residual boundary condition associated with some symmetric dissipative viscosity. See also the interesting recent investigations of Sueur [Su] in which he establishes that any (nonstrictly) dissipative boundary condition may be realized as the residual boundary condition associated with some (not necessarily symmetric) dissipative viscosity.*

# 5 The compressible Navier-Stokes and viscous MHD equations

In this section we present computations of $\mathcal{C}$ manifolds for some classical symmetric-dissipative systems with various boundary conditions, including standard inflow/outflow conditions for Navier-Stokes. For the small-amplitude case the results of sections 2.1 and 4 apply and give much information about when profiles satisfy the uniform Evans stability condition or transversality, and when the reduced hyperbolic boundary conditions, which are expressed in terms of $\mathcal{C}$ manifolds, satisfy the maximal dissipativity or uniform Lopatinski conditions. In a few cases we will say something about such properties for large amplitude profiles.

## 5.1 Isentropic Navier–Stokes equations

We start with the simplest case of noncharacteristic boundary layers for the isentropic compressible Navier–Stokes equations. In this case we are able to give a detailed description of possible boundary-layer connections, including large amplitude layers, and the resulting residual boundary conditions.

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**Computation of residual boundary conditions.** Consider the isentropic Navier–Stokes equations

$$\rho_t + (\rho u)_x + (\rho v)_y = 0, \tag{5.1}$$

$$(\rho u)_t + (\rho u^2)_x + (\rho uv)_y + P_x = (2\mu + \eta)u_{xx} + \mu u_{yy} + (\mu + \eta)v_{xy}, \tag{5.2}$$

$$(\rho v)_t + (\rho uv)_x + (\rho v^2)_y + P_y = \mu v_{xx} + (2\mu + \eta)v_{yy} + (\mu + \eta)u_{yx} \tag{5.3}$$

on the half-space $y > 0$, where $\rho$ is density, $u$ and $v$ are velocities in $x$ and $y$ directions, and $P = P(\rho)$ is pressure, and $\mu > |\eta| \geq 0$ are coefficients of first ("dynamic") and second viscosity. We assume a monotone pressure function

$$P'(\rho) > 0. \tag{5.4}$$

Seeking boundary-layer solutions $(\rho, u, v)(y)$, and setting $z := y$, we obtain profile equations

$$(5.5) \qquad \begin{aligned} &(\rho v)' = 0 \\ &(\rho uv)' = \mu u'' \\ &(\rho v^2)' + P(\rho)' = \epsilon v'', \end{aligned}$$

where "$'$" denotes $d/dz$, and $\epsilon := 2\mu + \eta$. Integrating $\int_{+\infty}^{z}$, we obtain

$$(5.6) \qquad \begin{aligned} &\rho v(z) = \rho_\infty v_\infty := m_\infty \\ &\mu u' = \rho uv - \rho_\infty u_\infty v_\infty \\ &\epsilon v' = \rho v^2 + P(\rho) - (\rho_\infty v_\infty^2 + P(\rho_\infty), \end{aligned}$$

where $(\rho_\infty, u_\infty, v_\infty) := (\rho, u, v)(+\infty)$ and $m := \rho v$ denotes momentum in the normal ($z$) direction. Within the set of allowable boundary conditions at $z = 0$ in our abstract framework, we consider a subset consisting of linear conditions

$$(5.7) \qquad \Gamma U(0) = g = (g^1, g^2, 0)$$

that are pure Dirichlet ($N'' = 0$) or else mixed Dirichlet–homogeneous Neumann boundary conditions ($N'' = N' = 2$), and which includes some of the most commonly used conditions.

With $\mathbb{R}^3_+ = \{(\rho_\infty, u_\infty, v_\infty) : \rho_\infty > 0\}$, $U(z) = (\rho(z), u(z), v(z))$, $U_\infty = (\rho_\infty, u_\infty, v_\infty)$, and $U_0 = (\rho_0, u_0, v_0)$, let

$$(5.8) \qquad \begin{aligned} \mathcal{C}_{\Gamma,g} = \{U_\infty \in \mathbb{R}^3_+ : &\text{ there exists } U(z) \text{ satisfying (5.6) together with} \\ &U(+\infty) = U_\infty \text{ and the boundary conditions (5.7) at } z = 0\ \}. \end{aligned}$$

For arbitrary $U_0$ with $\rho_0 > 0$ note that the constant profile

$$U(z) = U_0$$

determines an element $U_\infty = U_0 \in \mathcal{C}_{\Gamma,g}$, where $g = \Gamma U_0$. The goal here is to determine $\mathcal{C}_{\Gamma,g}$ in some (not necessarily small) neighborhood of $U_\infty = U_0$, and to understand how $\mathcal{C}_{\Gamma,g}$ changes with $g$. We consider separately the outflow and inflow case.

**Outflow with Dirichlet conditions.** We first consider the outflow case $v_0 < 0$, with Dirichlet boundary conditions. We have $\overline{A}_d^{11} < 0$ now, so $N_+^1 = 0$, $N_b = 2$ and thus we prescribe only the parabolic variables on the boundary:

$$(5.9) \qquad \Gamma U(0) = g = (u_0, v_0).$$

Noting that $m_\infty = m_0 < 0$ since $v_0 < 0$ and $\rho_0 > 0$, and rewriting the second equation in (5.6) as

$$(5.10) \qquad \mu u' = m_\infty (u - u_\infty),$$

we obtain solutions

$$u(z) = u_\infty \left(1 - (1 - \frac{u_0}{u_\infty})e^{\frac{m_\infty}{\mu} z}\right) \tag{5.11}$$

satisfying the conditions (on $u(z)$) in (5.8) with no restrictions so far except $\rho_0 > 0$, $u_\infty \in \mathbb{R}$.

Rewrite the third equation in (5.6) using the first equation to get

$$\begin{aligned} &\epsilon v' = m_\infty v + P\left(\frac{m_\infty}{v}\right) - \left(m_\infty v_\infty + P\left(\frac{m_\infty}{v_\infty}\right)\right) = \\ &m_\infty(v - v_\infty) + c^2(\rho, \rho_\infty)\left(\frac{m_\infty}{v} - \frac{m_\infty}{v_\infty}\right) = \\ &\left(m_\infty - \frac{m_\infty c^2(\rho, \rho_\infty)}{v v_\infty}\right)(v - v_\infty), \end{aligned} \tag{5.12}$$

where

$$c^2(\rho, \rho_\infty) := \frac{P(\rho) - P(\rho_\infty)}{\rho - \rho_\infty} > 0 \tag{5.13}$$

by assumption (5.4). Note that if we set

$$c_\infty = \sqrt{P'(\rho_\infty)} \tag{5.14}$$

we have $c(\rho_\infty, \rho_\infty) = c_\infty$.

With $(u_0, v_0)$ fixed, consider first a candidate state $U_\infty$ for membership in $\mathcal{C}_{\Gamma,g}$ satisfying

$$v_\infty < 0,\ v_\infty + c_\infty > 0, \text{ so } |v_\infty| < c_\infty. \tag{5.15}$$

For $z$ large we have $v \approx v_\infty$, $\rho \approx \rho_\infty$ and thus

$$m_\infty - \frac{m_\infty c^2(\rho, \rho_\infty)}{v v_\infty} > 0. \tag{5.16}$$

This implies that (5.12) has no nontrivial bounded solution with limit $v_\infty$ as $z \to +\infty$. In this case, the manifold $\mathcal{C}_{\Gamma,g}$ is an $N - N_+ = 2$ dimensional manifold defined by the single condition

$$v_\infty \equiv v_0, \tag{5.17}$$

which is the right number of conditions ($N_+ = 1$) for the Euler equations at a value $U_\infty$ satisfying (5.15). Indeed, this is the classical (specified-normal velocity) Euler boundary condition corresponding to the (specified-normal velocity, no-slip) Navier–Stokes boundary conditions $(u, v)(0) = (u_0, v_0)$.

In the case that

$$v_\infty + c_\infty < 0, \text{ so } |v_\infty| > c_\infty\ (N_+ = 0), \tag{5.18}$$

we obtain the opposite inequality

$$m_\infty - \frac{m_\infty c^2(\rho, \rho_\infty)}{v v_\infty} < 0. \tag{5.19}$$

Thus, in this case we have nontrivial solutions (5.12) with $v(\infty) = v_\infty$. Provided $v_0$ is close enough to $v_\infty$, $\mathcal{C}_{\Gamma,g}$ is a full neighborhood of $U_\infty$, as expected from the general theory. That is, there are no incoming hyperbolic characteristics and no boundary conditions imposed on the Euler equations in this case ($\dim \mathcal{C}_{\Gamma,g} = N - N_+ = 3$).

Uniform Evans stability of the above layers in the small amplitude case follows by Corollary 1.29(b).

Examining further, we see that connections to such states $U_\infty$ exist for boundary data $U_0$ such that

$$v_*^1 < v_0 \le v_\infty \text{ or } v_\infty \le v_0 < v_*^2, \tag{5.20}$$

where the $v_*^j$ are the nearest rest points of (5.12) to $v_\infty$. This is because all states $v$ in the closed interval between $v_0$ and $v_\infty$ then satisfy

$$m_\infty - \frac{m_\infty c^2(m_\infty/v, m_\infty/v_\infty)}{v v_\infty} < 0. \tag{5.21}$$

In the typical (genuinely nonlinear) case that $P''(\rho) > 0$, there is a single such rest state $v_*^2 > v_\infty$. The states $v_*^2$ and $v_\infty$ then correspond to endstates of stationary viscous shock waves on the whole line. Layers with $v < v_0$ are in this case "compressive", consisting of pieces of the viscous shock from $v_*^2$ to $v_\infty$, while layers with $v > v_0$ are "expansive" (decreasing $\rho$), analogous more to rarefactions; see [MN, CHNZ] for further discussion.

**Remarks 5.1.** *There are two transitions worth mentioning. One is when the number of hyperbolic characteristics changes; that is, at the boundary where inequality* (5.15) *changes its sense. Note that this transition has to do with the outer, hyperbolic solution* $U_\infty$ *and so we cannot deduce that such a transition occurs from knowledge of the Navier-Stokes boundary data* $U_0$ *alone, but must know the solution of the hyperbolic equation.*

*A second transition has to do with the "inner", boundary-layer structure, when* $U_0$ *goes out of range of* (5.20)*, with* $U_\infty$ *held fixed satisfying* (5.19)*. Consider a solution of the Navier-Stokes system with boundary data that includes states* $U_0$ *with some in and some out of range of* (5.20)*. In this case our description of the solution as boundary layer plus outer smooth solution fails, and it is a natural question to ask what happens instead. This question has been answered for the one-dimensional case in [MN]. The resolution is that in this case there is a more complicated structure at the boundary consisting of boundary layer plus shock or rarefaction waves incoming to the domain: that is, a (nonsmooth) boundary–Riemann solution.*

**Outflow with Neumann conditions**. We next consider the case of homogeneous Neumann conditions in the outflow case:

$$\Gamma U(0) = (u', v')(0) = (0, 0) = g. \tag{5.22}$$

Again consider a candidate state $U_\infty$ satisfying $v_\infty < 0$ for membership in $\mathcal{C}_{\Gamma,g}$. The vanishing of $u'(0)$ and $v'(0)$ implies that $u(0)$ and $v(0)$ are rest points of (5.10) and (5.12) respectively. Thus $u(z) \equiv u_\infty$, $v(z) \equiv v_\infty$, so $U_\infty$ can only be the endstate of a constant layer, and

$$\mathcal{C}_{\Gamma,g} = \{U_\infty \in \mathbb{R}^3 : \rho_\infty > 0, v_\infty < 0\}. \tag{5.23}$$

In the case (5.15) we have $N_+ = N_-^2 = 1$, so $N'' = 2 > N_-^2$; thus, by Corollary 2.5 these layers are not transversal, and thus are not even low frequency Evans stable by Lemma 1.23. Observe that the "correct" dimension for $\mathcal{C}_{\Gamma,g}$ in this case is $N - N_+ = 2$.

In the case when $U_\infty$ satisfies (5.18), we have $N_+ = 0$, so $N'' = N_-^2 = 2$ and $\mathcal{C}_{\Gamma,g}$ has the right dimension. As expected there are no Euler boundary conditions and by Corollary 1.29(b), the constant layers are Evans stable.

**Inflow with Dirichlet conditions.** In the case of inflow $v_0 > 0$, so $N_+^1 = 1$, $N_+ = 3$ and we prescribe

$$\Gamma U(0) = (\rho_0, u_0, v_0) = g. \tag{5.24}$$

Let $U_\infty$ be a candidate state for $\mathcal{C}_{\Gamma,g}$ with $v_\infty > 0$. The equation for the transverse velocity $u$ decouples as (5.10); however, due to the opposite sign $m_\infty > 0$, this equation now has no nonconstant solutions converging to $u_\infty$. Hence, the transverse velocity is specified as

$$u_\infty = u_0.$$

Continuing as before, we find two cases, according as $v_\infty - c_\infty \gtrless 0$, i.e.,

$$v_\infty \gtrless c_\infty.$$

In the first case $N_+ = 3$ and we find as before that there are no nontrivial solutions of (5.12), whence $v_\infty = v_0$, and $\rho_\infty = m_\infty/v_\infty = m_0/v_0 = \rho_0$. Thus, only the constant layer is possible, and the induced hyperbolic boundary conditions are full Dirichlet,

$$U_\infty = U_0, \;\; \mathcal{C}_{\Gamma,g} = \{U_0\}$$

in agreement with the classical Euler conditions.

In the second case $N_+ = 2$ and there exist nontrivial connections on the range specified by (5.21). The induced hyperbolic boundary conditions defining $\mathcal{C}_{\Gamma,g}$ are

$$u_\infty = u_0, \quad m_\infty = m_0,$$

specifying transverse velocity and *momentum*, an interesting variation on (5.17) in the outflow case.

Uniform Evans stability of the above layers in the small amplitude case follows from Corollary 1.29(b).

**Inflow with mixed boundary conditions.** Finally, we consider the case of mixed Dirichlet-homogeneous Neumann conditions

$$\Gamma U(0) = (\rho(0), u'(0), v'(0)) = (\rho_0, 0, 0) = g \tag{5.25}$$

for the inflow case. Again, $u(z) \equiv u_\infty$ and $v(z) \equiv u_\infty$, since $u(0)$ and $v(0)$ are rest points of (5.10) and (5.12) respectively. The mass equation then implies $\rho(z) \equiv \rho_\infty$, so in particular $\rho_\infty = \rho_0$. Therefore, a candidate state $U_\infty$ can belong to $\mathcal{C}_{\Gamma,g}$ only if it is the endpoint of a constant layer, and we have

$$\mathcal{C}_{\Gamma,g} = \{U_\infty \in \mathbb{R}^3 : v_\infty > 0, \rho_\infty = \rho_0\}, \quad \dim \mathcal{C}_{\Gamma,g} = 2. \tag{5.26}$$

Consider the cases

$$v_\infty - c_\infty \gtrless 0, \tag{5.27}$$

and recall that $N_b = 3 = N_+ + N_-^2$. In these cases $N_+ = 3$, $N_-^2 = 0$ and $N_+ = 2$, $N_-^2 = 1$ respectively, so in both cases $N'' = 2 > N_-^2$. Corollary 2.5 implies that these constant layers are never transversal; hence they fail to satisfy even low frequency Evans stability. Observe that in both cases $\mathcal{C}_{\Gamma,g}$ fails to have the "correct" dimension $N - N_+$.

### 5.1.1 Maximal dissipativity/uniform Lopatinski condition.

The question of uniform Evans stability for large amplitude layers remains a mostly open question. For the one-dimensional isentropic case with $\gamma$-law equation of state, it has been shown numerically for Dirichlet boundary conditions that noncharacteristic boundary layers are stable, independent of amplitude [CHNZ]. For the full, nonisentropic case on the other hand, it has been shown for Dirichlet boundary conditions that, even in one dimension and for $\gamma$-law equation of state, instabilities may occur [SZ]. Here we show that the residual boundary conditions determined in the previous subsection are maximally dissipative (and thus satisfy the uniform Lopatinski condition) for all amplitudes. When $\Gamma$ is a full Dirichlet condition ($N'' = 0$), this conclusion follows without any computation, since the residual boundary conditions by the analysis of the previous subsection have form independent of the amplitude of the boundary layer, and for small amplitudes are known (Theorem 4.4) to be maximally dissipative. We carry out the computations nonetheless, to show how they work out in this simplest case. See [SZ] for one-dimensional calculations involving more general, not necessarily decoupled, boundary conditions.

**1. Dirichlet conditions, outflow, $v_\infty + c_\infty > 0 > v_\infty$.** The hyperbolic problem is symmetric, with one incoming, and two outgoing characteristics. Rewriting the isentropic Euler equations in $U := (\rho, u, v)$ coordinates and linearizing about $(\rho_\infty, u_\infty, v_\infty)$, we obtain $U_t + A^1 U_x + A^2 U_y$, where

$$A_2 = \begin{pmatrix} v_\infty & 0 & \rho_\infty \\ 0 & v_\infty & 0 \\ c_\infty^2/\rho_\infty & 0 & v_\infty \end{pmatrix},$$

while the residual boundary condition (linear, in this case) is

$$\Gamma_{res} U = g, \qquad \Gamma = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix},$$

hence the kernel of the linearized boundary condition is the set of vectors $(a, b, 0)$. Applying the positive definite symmetrizer $S = \mathrm{diag}\{c_\infty^2/\rho_\infty^2, 1, 1\}$, we obtain

$$SA_2 = \begin{pmatrix} v_\infty c_\infty^2/\rho_\infty^2 & 0 & c_\infty^2/\rho_\infty \\ 0 & v_\infty & 0 \\ c_\infty^2/\rho_\infty & 0 & v_\infty \end{pmatrix}, \tag{5.28}$$

which is evidently negative definite on the kernel of $\Gamma_{res}$; hence $\Gamma_{res}$ is maximally dissipative.

**2. Dirichlet conditions, outflow, $v_\infty + c_\infty < 0$.** In this case $SA_2$ is negative definite, so the residual boundary conditions are automatically *maximally dissipative*.

**3. Dirichlet conditions, inflow, $v_\infty - c_\infty > 0$.** Again, this is a trivial case, in which all hyperbolic modes are incoming, and the residual boundary conditions are $\Gamma_{res} U = U$. So $\ker \Gamma_{res} = \{0\}$, and thus $\Gamma_{res}$ is maximally dissipative.

**4. Dirichlet conditions, inflow, $v_\infty - c_\infty < 0 < v_\infty$.** In this final case, we have prescription of momentum $m_\infty$ and transverse velocity $u_\infty$, and the kernel of the linearized boundary condition is spanned by $w := (\rho_\infty, 0, -v_\infty)$. Computing

$$w^T SA_2 w = v_\infty (v_\infty^2 - c_\infty^2) < 0,$$

we find that the restriction of $SA_2$ to the kernel is again negative definite, so that the residual boundary condition is indeed maximally dissipative in agreement with the abstract theory.

**Remark 5.2.** *1. We remark on the contrast between specification of velocity vs. momentum in outgoing vs. incoming case. (Specifying $\rho$ or $v$ along with $u$ in the incoming case wouldn't be dissipative in general.)*

*2. **Neumann or mixed conditions.** The only case in section 5.1 involving Neumann conditions that was Evans stable was the outflow case when $v_\infty + c_\infty < 0$. Again, $SA_2$ is negative definite so the residual conditions are maximally dissipative. In the remaining three cases involving Neumann or mixed conditions, maximal dissipativity fails. For example, in the inflow case ($v_\infty > 0$) $\Gamma_{res} = (1, 0, 0)$, and so $SA_2$ is positive definite on* $\ker \Gamma_{res}$.

### 5.1.2 Transversality of large amplitude layers

We now show how to verify that transversality holds in the three cases where we constructed large amplitude layers:

(a) outflow/Dirichlet/$v_\infty + c_\infty > 0$;
(b) outflow/Dirichlet/$v_\infty + c_\infty < 0$;
(c) inflow/Dirichlet/$v_\infty - c_\infty < 0$.

It is helpful first to reformulate the definition of transversality geometrically. Let $w(z)$ be a possibly large amplitude layer profile converging to $q_\infty := (\rho_\infty, u_\infty, v_\infty)$. Set

$$W_0 := (w(0), w_z^2(0)),\ W_\infty = (q_\infty, 0) \tag{5.29}$$

and observe that $W_0 \in \mathbb{R}^5$ lies in both the stable manifold of $W_\infty$, denoted $\mathcal{W}^s(W_\infty)$, and the center-stable manifold of $W_\infty$, denoted $\mathcal{W}^{cs}(W_\infty)$. The conditions defining transversality of $w(z)$ in Definition 1.11 can be rephrased:

(i) $\Gamma : T_{W_0}(\mathcal{W}^s(W_\infty)) \to \mathbb{R}^{N_b}$ is injective,
(ii) $\Gamma : T_{W_0}(\mathcal{W}^{cs}(W_\infty)) \to \mathbb{R}^{N_b}$ is surjective.

These may be recognized as the conditions that $\Upsilon$ be full rank on the stable and center–stable manifolds, respectively, of $W_\infty$, which correspond by definition to the following geometric version of transversality.

**Lemma 5.3** (Geometric transversality conditions)**.** *The transversality conditions of Definition 1.11 are equivalent to the conditions that*

*(i') the level set $\{W : \Upsilon(W) = \Upsilon(W_0)\}$ meets the stable manifold of $W_\infty$ transversally in phase space $W = (w, w_z^2)$.*

*(ii') the level set $\{W : \Upsilon(W) = \Upsilon(W_0)\}$ meets the center-stable manifold of $W_\infty$ transversally in phase space $W = (w, w_z^2)$.*

**Remark 5.4.** *The use of the word "transversal" in (i') is perhaps nonstandard; it is used to suggest a minimal intersection. We mean $\{W : \Upsilon(W) = \Upsilon(W_0)\} \cap \mathcal{W}^s(W_\infty) = \{W_0\}$; that is, the intersection is a single point.*

The stable and center-stable manifolds are easily parametrized in each of cases (a), (b), and (c) above, so conditions (i) and (ii) can be checked explicitly. For example, consider case (c), where $N_b = N_+ + N_-^2 = 2 + 1 = 3$, the viscous boundary condition is

$$\Gamma U(0) = (\rho_0, u_0, v_0) := U_0 \tag{5.30}$$

with $U_0$ fixed, $w(0) = U_0$, $w(\infty) = q_\infty$ is fixed, and the dimensions of the stable and center-stable manifolds above are 1 and 4 respectively. Recalling the induced hyperbolic boundary conditions

$$u_\infty = u_0,\ \ \rho_\infty v_\infty = \rho_0 v_0, \tag{5.31}$$

we see that those manifolds can be parametrized as follows:

$$\mathcal{W}^s(W_\infty) = \left\{ \begin{pmatrix} \frac{\rho_\infty v_\infty}{a} \\ u_\infty \\ a \\ 0 \\ f(a, \rho_\infty, v_\infty) \end{pmatrix} : a \in A \right\}$$

$$\mathcal{W}^{cs}(W_\infty) = \left\{ \begin{pmatrix} \frac{bd}{a} \\ c \\ a \\ 0 \\ f(a, b, d) \end{pmatrix} : a \in A \subset \mathbb{R}, (b, c, d) \in B \subset \mathbb{R}^3 \right\} \tag{5.32}$$

where $B$ is a neighborhood of $q_\infty$, $A$ is a neighborhood of $v_0$ determined by (5.20), and the function $f$ (whose form turns out not to be important for verifying transversality) can be read off from (5.12). Independent vectors spanning the tangent spaces to $\mathcal{W}_s(W_\infty)$ (resp. $\mathcal{W}^{cs}(W_\infty)$) at $W_0$ may be computed by differentiating the formulas (5.32) with respect to $a$ (resp. $a$,$b$, $c$, $d$). With these explicit formulas the injectivity and surjectivity conditions above are immediately obvious.

We summarize this discussion in the following

**Proposition 5.5.** *The isentropic Navier-Stokes large-amplitude layer profiles described in cases (a), (b), (c) at the beginning of this section are transversal.*

## 5.2 Full Navier-Stokes equations

Next we consider the full (nonisentropic) Navier–Stokes equations

$$\begin{aligned}
&(a)\ \rho_t + (\rho u)_x + (\rho v)_y = 0, \\
&(b)\ (\rho u)_t + (\rho u^2)_x + (\rho u v)_y + p_x = (2\mu + \eta) u_{xx} + \mu u_{yy} + (\mu + \eta) v_{xy}, \\
&(c)\ (\rho v)_t + (\rho u v)_x + (\rho v^2)_y + p_y = \mu v_{xx} + (2\mu + \eta) v_{yy} + (\mu + \eta) u_{yx}, \\
&(d)\ (\rho E)_t + (u \rho E)_x + (v \rho E)_y + (pu)_x + (pv)_y = \\
&\qquad \kappa T_{xx} + \kappa T_{yy} + \Big((2\mu + \eta) u u_x + \mu v (v_x + u_y) + \eta u v_y\Big)_x + \\
&\qquad\qquad \Big((2\mu + \eta) v v_y + \mu u (v_x + u_y) + \eta v u_x\Big)_y
\end{aligned} \tag{5.33}$$

where $\rho$ is density, $u$ and $v$ are velocities in the $x$ and $y$ directions, $p$ is pressure, and $e$ and $E = e + \frac{u^2}{2} + \frac{v^2}{2}$ are specific internal and total energy respectively. The constants $\mu > |\eta| \geq 0$ and $\kappa > 0$ are coefficients of first ("dynamic") and second viscosity and heat conductivity. Finally, $T$ is the temperature and we assume that the internal energy $e$ and the pressure $p$ are known functions of density and temperature:

$$p = p(\rho, T), \quad e = e(\rho, T).$$

It is clear for full (nonisentropic) gas dynamics that the uniform Lopatinski condition does *not* hold for arbitrary amplitudes, even for the simplest case of a $\gamma$-law gas. The proof of one-dimensional ($\eta = 0$) viscous instability in [SZ] for compressive boundary-layers of such a gas [3] was based on showing that the Lopatinski determinant (a multiple of $\lambda$) restricted to the positive real axis could change sign as parameters, including amplitude, were varied; in particular, the determinant could vanish, yielding hyperbolic instability. So we cannot hope to show that the residual boundary conditions independent of amplitude as before. A brief examination reveals, likewise, that the residual boundary conditions are rather complicated to describe: in short, there appears to be no hope of other than a *local* analysis near the limiting constant layer. We carry this out below, and compute the linearized boundary condition for the linearized hyperbolic problem at a constant layer.

The profile equations are, setting $\nu := 2\mu + \eta$:

$$\begin{aligned}(\rho v)' &= 0\\ (\rho u v)' &= \mu u''\\ (\rho v^2)' + p' &= \nu v''\\ \big(\rho v(e + \frac{u^2+v^2}{2}) + pv\big)' &= \kappa T'' + \frac{1}{2}(\mu u^2 + \nu v^2)''\end{aligned}$$

Let us introduce the unknown $m = \rho v$. Writing the equations in terms of the unknowns $(m, u, v, T)$ and integrating once gives the following equations with $p = P(v, T)$ and $p_\infty = P(v_\infty, T_\infty)$:

$$\begin{aligned}(a)\ m &= m_\infty\\ (b)\ u' &= \frac{m_\infty}{\mu}(u - u_\infty)\\ (c)\ v' &= \frac{m_\infty}{\nu}(v - v_\infty) + \frac{p - p_\infty}{\nu}\\ (d)\ T' &= \frac{m_\infty}{\kappa}(e - e_\infty) - \frac{m_\infty}{2\kappa}(u - u_\infty)^2\\ &\quad -\frac{m_\infty}{2\kappa}(v - v_\infty)^2 + \frac{p_\infty}{\kappa}(v - v_\infty)\end{aligned} \tag{5.34}$$

The equation for $X = (u, v, T)$ reads $X' = F(X; m_\infty, X_\infty)$ where $X_\infty$ and $m_\infty$ are parameters.

**Case of an outgoing flow.** Let $U = (m, u, v, T) = (m, X)$ and consider an outgoing flow, $v < 0$, so $N_b = N' + N_+^1 = 3 + 0 = 3$. We fix a state $U_0 \in \mathcal{U}_\partial$ for which $v_0 < 0$ consider boundary conditions that are just Dirichlet conditions on $X$:

$$\Gamma U(0) = X(0) = X_0. \tag{5.35}$$

[3] Specifically, for $\gamma > 2$ and $2\mu + \eta > \kappa$, in the inflow case $v > 0 > v - c_\infty$ with full Dirichlet conditions imposed at the boundary, instability was shown for layers consisting of a sufficiently large portion of a sufficiently large-amplitude 1-shock.

As usual we define $\mathcal{C}_{\Gamma,X_0}$ as the set of states $(m_\infty, X_\infty)$ such that there exists a solution $(m, X)$ of the equations (5.34) on $[0, +\infty[$, such that $X(0) = X_0$ and $U(\infty) = (m_\infty, X_\infty)$. Clearly, $U_0 \in \mathcal{C}_{\Gamma,X_0}$, and near $U_0$ we can write $\mathcal{C}_{\Gamma,X_0}$ as the union

$$\mathcal{C}_{\Gamma,X_0} = \cup_{m_\infty} \mathcal{C}_{m_\infty,X_0} \tag{5.36}$$

where $\mathcal{C}_{m_\infty,X_0}$ is the set of $X_\infty \in \mathbb{R}^3$ such that there exists a solution to equations 5.34 (b-c) on $[0, +\infty[$ satisfying $X(0) = X_0$ and $X(\infty) = X_\infty$. Corollary 2.5 shows that the constant layer $U(z) = U_0$ is transversal, and thus $\mathcal{C}_{\Gamma,X_0}$ is a $\mathcal{C}^\infty$ manifold near $U_0$. By the argument of [Se], Lemma 15.2.5 (or alternatively, by an argument similar to our proof of Lemma 4.6), the tangent space $T_{U_0}\mathcal{C}_{\Gamma,X_0}$ is given by

$$T_{U_0}\mathcal{C}_{\Gamma,X_0} = \mathbb{R} \times \mathbb{E}^-\big(D_X F(X_0; m_0, X_0)\big). \tag{5.37}$$

Hence we are lead to calculate the stable invariant subspace of $D_X F(X_0; m_0, X_0)$. One finds

$$D_X F(X; m, X) = \begin{pmatrix} \frac{m}{\mu} & 0 & 0 \\ 0 & \frac{1}{\nu}(m + P'_v) & \frac{1}{\nu}P'_T \\ 0 & \frac{1}{\kappa}(P_\infty + m e'_v) & \frac{m}{\kappa}e'_T \end{pmatrix} \tag{5.38}$$

Since $m_0 < 0$, the matrix $D_X F(X_0; m_0, X_0)$ has at least one negative eigenvalue which is $m_0/\mu$. It has exactly two eigenvalues with negative real part (in fact real negative eigenvalues) if and only if

$$\det \begin{bmatrix} \frac{1}{\nu}(m + P'_v) & \frac{1}{\nu}P'_T \\ \frac{1}{\kappa}(P_\infty + m e'_v) & \frac{m}{\kappa}e'_T \end{bmatrix} < 0. \tag{5.39}$$

In that case $\mathbb{E}^-\big(D_X F(X_0; m_0, X_0)\big)$ has dimension 2 and $T_{U_0}\mathcal{C}_{\Gamma,X_0}$ has dimension 3. Let us call $\lambda_-$ the second negative eigenvalue of $D_X F(X_0; m_0, X_0)$. Then, the tangent space to $\mathcal{C}_{\Gamma,X_0}$ at the point $(m_0, X_0)$ is defined by the equations:

$$(\dot m, \dot u, \dot v, \dot T) \in T_{(m_0,X_0)}\mathcal{C}_{\Gamma,X_0} \iff \big(\tfrac{1}{\nu}(m_0 + P'_v) - \lambda_-\big)\, \dot v + \tfrac{1}{\nu}P'_T \dot T = 0. \tag{5.40}$$

We do not know whether these boundary conditions have already appeared in the theory of the Euler equations, or if they have a special physical meaning. As we already know from the general theory, they are maximally dissipative for the Euler equations.

## 5.3 MHD equations

The equations of isentropic magnetohydrodynamics (MHD) are

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0 \\ \partial_t(\rho u) + \mathrm{div}(\rho u^t u) + \nabla p + H \times \mathrm{curl} H = \varepsilon\mu\Delta u + \varepsilon(\mu + \eta)\nabla \mathrm{div} u \\ \partial_t H + \mathrm{curl}(H \times u) = \varepsilon\sigma\Delta H \end{cases} \tag{5.41}$$

$$\text{div} H = 0, \tag{5.42}$$

where $\rho \in \mathbb{R}$ represents density, $u \in \mathbb{R}^3$ fluid velocity, $p = p(\rho) \in \mathbb{R}$ pressure, and $H \in \mathbb{R}^3$ magnetic field, with viscosities $\mu > |\eta| \geq 0$ and magnetic resistivity $\sigma \geq 0$; *for simplicity, we consider here the case* $\sigma = 0$. When $H \equiv 0$ (5.41) reduces to the equations of isentropic fluid dynamics.

The equations (5.41) are not yet in a form that satisfies the assumptions of section 1.1. For example, the noncharacteristic condition is violated for every state $(\rho, u, H)$. The equations may be put in conservative form using identity

$$H \times \text{curl} H = (1/2)\text{div}(|H|^2 I - 2H^t H)^{\text{tr}} + H \text{div} H \tag{5.43}$$

together with constraint (5.42) to express the second equation as

$$\partial_t(\rho u) + \text{div}(\rho u^t u) + \nabla p + (1/2)\text{div}(|H|^2 I - 2H^t H)^{\text{tr}} = \varepsilon \mu \Delta u + \varepsilon(\mu + \eta)\nabla \text{div} u. \tag{5.44}$$

They may be put in symmetrizable (but no longer conservative) form by a further change, using identity

$$\text{curl}(H \times u) = (\text{div} u)H + (u \cdot \nabla)H - (\text{div} H)u - (H \cdot \nabla)u \tag{5.45}$$

together with constraint (5.42) to express the third equation as

$$\partial_t H + (\text{div} u)H + (u \cdot \nabla)H - (H \cdot \nabla)u = \sigma \varepsilon \Delta H. \tag{5.46}$$

Forgetting the constraint equation, we get a $7 \times 7$ symmetric system that satisfies all our structural assumptions except for (H4).

**Remark 5.6.** *1) Define*

$$c^2 = p'(\rho) > 0, \quad v = H/\sqrt{\rho}, \quad b = |\hat{\xi} \times v|, \quad \hat{\xi} = \xi/|\xi|, \tag{5.47}$$

$$c_f^2 := \frac{1}{2}\Big(c^2 + |v|^2 + \sqrt{(c^2 - |v|^2)^2 + 4b^2 c^2}\Big),$$
$$c_s^2 := \frac{1}{2}\Big(c^2 + |v|^2 - \sqrt{(c^2 - |v|^2)^2 + 4b^2 c^2}\Big).$$

*The boundary* $x_3 = 0$ *is noncharacteristic for the hyperbolic part when*

$$u_3 \notin \{0, \pm v_3, \pm c_s(n), \pm c_f(n)\}, \tag{5.48}$$

*where* $c_s(n)$ *and* $c_f(n)$ *are the slow and fast speeds computed in the normal direction* $n = (0, 0, 1)$. *Lemma 8.2 of [GMWZ6] shows that if we assume in addition*

$$0 < |v| \neq c, \ \ |u_3| > |v_3|, \tag{5.49}$$

*then Hypothesis (H4′) is satisfied. For the stability analysis, which requires the construction of* $K$*-families of symmetrizers ([GMWZ6], Definition 3.5), we must use* (5.44), (5.46). *For* *the purposes of deriving profile equations, describing $\mathcal{C}$ manifolds, and computing linearized residual boundary conditions, the form (5.41), (5.42) is more convenient to use.*

*2) As the divergence-free condition is preserved by the evolution of the equations, we are free to ignore it in establishing stability of solutions. Results on instability, however, must be examined to check whether associated unstable modes are true, divergence-free instabilities or only apparent, spurious instabilities.*

With $\nu := 2\mu + \eta$ the profile equations are:

$$\begin{aligned} (\rho u_3)' &= 0 \\ (\rho u_1 u_3 - H_1 H_3)' &= \mu u_1'' \\ (\rho u_2 u_3 - H_2 H_3)' &= \mu u_2'' \\ \big(\rho u_3^2 + p + \frac{1}{2}(H_1^2 + H_2^2 - H_3^2)\big)' &= \nu u_3'' \\ (H_1 u_3 - u_1 H_3)' &= 0 \\ (H_2 u_3 - u_2 H_3)' &= 0 \\ H_3' &= 0 \end{aligned} \tag{5.50}$$

By choosing the set of new unknowns $m := \rho u_3$, $\alpha := (H_1 u_3 - u_1 H_3)$, $\beta := (H_2 u_3 - u_2 H_3)$, $H_3$, $u_1$, $u_2$, $u_3$ and integrating once, the system reduces to a system on $u = (u_1, u_2, u_3)$, where $M := (m, \alpha, \beta, H_3) \in \mathbb{R}^4$ and $u_\infty = (u_{1\infty}, u_{2\infty}, u_{3\infty}) \in \mathbb{R}^3$ are parameters:

$$\begin{aligned} M &= M_\infty \\ u_1' &= \frac{m}{\mu}(u_1 - u_{1\infty}) - \frac{\alpha H_3}{\mu}\Big(\frac{1}{u_3} - \frac{1}{u_{3\infty}}\Big) - \frac{H_3^2}{\mu}\Big(\frac{u_1}{u_3} - \frac{u_{1\infty}}{u_{3\infty}}\Big) \\ u_2' &= \frac{m}{\mu}(u_2 - u_{2\infty}) - \frac{\beta H_3}{\mu}\Big(\frac{1}{u_3} - \frac{1}{u_{3\infty}}\Big) - \frac{H_3^2}{\mu}\Big(\frac{u_2}{u_3} - \frac{u_{2\infty}}{u_{3\infty}}\Big) \\ u_3' &= \frac{m}{\nu}(u_3 - u_{3\infty}) + \frac{1}{\nu}\Big(p\big(\frac{m}{u_3}\big) - p\big(\frac{m}{u_{3\infty}}\big)\Big) + \\ &+ \frac{1}{2\nu}(H_1^2 - H_{1\infty}^2) + \frac{1}{2\nu}(H_2^2 - H_{2\infty}^2) - \frac{1}{2\nu}(H_3^2 - H_{3\infty}^2). \end{aligned} \tag{5.51}$$

In the last equation it is understood that $H_1 = \frac{\alpha + u_1 H_3}{u_3}$ and $H_2 = \frac{\beta + u_2 H_3}{u_3}$. As in the calculations of section 5.2, the system has the form $u' = F(u; M, u_\infty)$. The Jacobian matrix is

$$D_u F(u; M, u) = \begin{bmatrix} a & 0 & e/\mu \\ 0 & a & d/\mu \\ e/\mu & d/\nu & b \end{bmatrix},$$

where $M := (m, \alpha, \beta, H_3)$ and

$$\begin{aligned} a &:= \tfrac{m}{\mu} - \tfrac{H_3^2}{\mu u_3} \qquad & b &:= \tfrac{m}{\nu}\big(1 - \tfrac{P'(\rho)}{u_3^2}\big) - \tfrac{H_1^2 + H_2^2}{\nu u_3} \\ e &:= \tfrac{H_1 H_3}{u_3} \qquad & d &:= \tfrac{H_2 H_3}{u_3}. \end{aligned} \tag{5.52}$$

The eigenvalues of $D_uF(u; M, u)$ are

$$a \text{ and } \lambda_\pm := \frac{a+b}{2} \pm \sqrt{\frac{(a-b)^2}{4} + \frac{e^2+d^2}{\mu\nu}}, \tag{5.53}$$

and we have

$$\lambda_+\lambda_- = ab - (e^2+d^2)/\mu\nu. \tag{5.54}$$

**Case of an outgoing flow.** Let $U = (m, \alpha, \beta, H_3, u) = (M, u)$ and fix a state $U_0 = (M_0, u_0) \in \mathcal{U}_\partial$ which satisfies the conditions (5.48) and (5.49) and for which the flow is outgoing, $u_3 < 0$, so $N_b = N' + N_+^1 = 3 + 0 = 3$. We consider boundary conditions that are just Dirichlet conditions on $u$:

$$\Gamma U(0) = u(0) = u_0. \tag{5.55}$$

As usual we define $\mathcal{C}_{\Gamma,u_0}$ as the set of states $(M_\infty, u_\infty)$ such that there exists a solution $(M, u)$ of the equations (5.51) on $[0, +\infty[$, such that $u(0) = u_0$ and $U(\infty) = (M_\infty, u_\infty)$. Clearly, $U_0 \in \mathcal{C}_{\Gamma,u_0}$, and near $U_0$ we can write $\mathcal{C}_{\Gamma,u_0}$ as the union

$$\mathcal{C}_{\Gamma,u_0} = \cup_{M_\infty} \mathcal{C}_{M_\infty,u_0}, \tag{5.56}$$

where $\mathcal{C}_{M_\infty,u_0}$ is the set of $u_\infty \in \mathbb{R}^3$ such that there exists a solution to the last three equations in (5.51) on $[0, +\infty[$ satisfying $u(0) = u_0$ and $u(\infty) = u_\infty$. Corollary 2.5 shows that the constant layer $U(z) = U_0$ is transversal, and thus $\mathcal{C}_{\Gamma,X_0}$ is a $\mathcal{C}^\infty$ manifold near $U_0$. By the argument of section 5.2, the tangent space $T_{U_0}\mathcal{C}_{\Gamma,u_0}$ is given by

$$T_{U_0}\mathcal{C}_{\Gamma,u_0} = \mathbb{R}^4 \times \mathbb{E}^-\big(D_uF(u_0; M_0, X_0)\big). \tag{5.57}$$

We proceed to compute the stable subspace of $\mathcal{F}_0 := D_uF(u_0; M_0, X_0)$.

The condition (5.49) implies that $\mathcal{F}_0$ has at least one negative eigenvalue, namely $a$, with corresponding eigenvector $(-\frac{d}{\nu}, \frac{e}{\mu}, 0)$. The matrix $\mathcal{F}_0$ will have exactly one, two, or three negative eigenvalues depending on whether

$$\lambda_- > 0,\ \lambda_- < 0 < \lambda_+, \ \text{ or } \lambda_+ < 0 \tag{5.58}$$

respectively. From (5.57) we see that the corresponding dimensions of $\mathcal{C}_{\Gamma,u_0}$ are 5, 6, and 7 respectively. By Proposition 2.4 the dimension of $\mathcal{C}_{\Gamma,u_0}$ is $N - N_+$, where $N_+$ is the number of positive eigenvalues of $\overline{A}_3$. The eigenvalues of $\overline{A}_3$ are

$$\lambda_0 = u_3,\ \lambda_{\pm s} = u_3 \pm c_s(n),\ \lambda_{\pm 2} = u_3 \pm v_3,\ \lambda_{\pm f} = u_3 \pm c_f(n), \tag{5.59}$$

so we observe that the cases (5.58) correspond to the cases

$$|u_3| < c_s(n),\ c_s(n) < |u_3| < c_f(n),\ c_f(n) < |u_3| \tag{5.60}$$

respectively. For example, consider the case $\lambda_- < 0 < \lambda_+$, which by formula (5.54) occurs if and only if

$$ab - \frac{e^2 + d^2}{\mu\nu} < 0. \tag{5.61}$$

A short computation now shows that the stable subspace of $\mathcal{F}_0$ is spanned by

$$\begin{aligned} &\{(-\frac{d}{\nu}, \frac{e}{\mu}, 0), (\frac{e}{\mu}, \frac{d}{\mu}, \lambda_- - a)\}, \\ &\text{where } \lambda_- - a = \frac{b-a}{2} - \sqrt{\frac{(a-b)^2}{4} + \frac{e^2+d^2}{\mu\nu}} \end{aligned} \tag{5.62}$$

The linearized residual boundary condition is thus expressed by

$$(\dot{M}, \dot{u}) \in T_{U_0}\mathcal{C}_{\Gamma,u_0} \Leftrightarrow \dot{u} \cdot \left( \frac{e}{\mu}(a - \lambda_-), \frac{d}{\nu}(a - \lambda_-), \frac{d^2}{\mu\nu} + \frac{e^2}{\mu^2} \right) = 0, \tag{5.63}$$

where $a$,$b$,$e$,$d$ are evaluated at $U_0$.

**Remark 5.7.** *The hypotheses and conclusions of Theorem 1.30 on Evans stability and the existence of small viscosity limits apply to small amplitude layers for all the physical examples considered in section 5 except*

*a)isentropic NS/inflow ($v_0 > 0$)/mixed Dirichlet-Neumann boundary conditions* (5.25)

*b)isentropic NS/outflow ($v_0 < 0$)/$v_\infty + c_\infty > 0$/Neumann boundary conditions* (5.22).

*Additional examples for the full Navier Stokes and viscous MHD equations where Theorem 1.30 applies can be deduced from Corollary 1.29(b).*

# A Construction of Approximate solutions

In this appendix, we give the construction of approximate solutions, following an approach similar to that used in [GG, GMWZ7]. A new feature here is that we obtain global approximate solutions on a domain $\Omega$ with compact closure and smooth boundary. The same construction works on unbounded domains whose boundary coincides with a half-space outside a compact set.

We seek high-order approximate solutions to

$$\begin{aligned} &(a)\mathcal{L}_\varepsilon(u) := A_0(u)u_t + \sum_{j=1}^{d} A_j(u)\partial_j u - \varepsilon \sum_{j,k=1}^{d} \partial_j\big(B_{jk}(u)\partial_k u\big) = 0, \\ &(b)\Upsilon(u, \partial_T u^2, \partial_\nu u^2) = (g_1, g_2, 0) \text{ on } \partial\Omega \end{aligned} \tag{A.1}$$

which converge to a given solution $u^0(t,x)$ of the inviscid hyperbolic problem:

$$\text{(A.2)}\qquad \begin{aligned}&\mathcal{L}_0(u^0)=0 \text{ on } [-T_0,T_0]\times\Omega\\ &u^0(t,x_0)\in\mathcal{C}(t,x_0) \text{ for } (t,x_0)\in[-T_0,T_0]\times\partial\Omega,\end{aligned}$$

where $\mathcal{C}(t,x_0)$ is the endstate manifold defined in Assumption 1.12 (see also Prop. 2.6). Using the cutoff $\chi(x)$ and the normal coordinates $(x_0,z)$ in a collar neighborhood of $\partial\Omega$ defined in section 1.4, we look for an approximate solution of the form

$$\text{(A.3)}\qquad \begin{aligned}&u_a(t,x)=\sum_{0\le j\le M}\varepsilon^j\mathcal{U}^j(t,x,\frac{z}{\epsilon})+\epsilon^{M+1}u^{M+1}(t,x),\\ &\mathcal{U}^j(t,x,\frac{z}{\epsilon})=\chi(x)V^j(t,x_0,\frac{z}{\epsilon})+u^j(t,x).\end{aligned}$$

Here $u^0$ satisfies(A.2) and $V^0$ is given by

$$\text{(A.4)}\qquad V^0(t,x_0,Z)=W(Z,t,x_0,u^0(t,x_0))-u^0(t,x_0),$$

for a profile $W(Z,t,x_0,u^0(t,x_0))$ as in Assumption 1.12.

The $V^j_\pm(Z,x_0,t)$ are boundary layer profiles constructed to be exponentially decreasing to 0 as $Z\to\pm\infty$. For the moment we just assume enough regularity so that all the operations involved in the construction make sense. A precise statement is given in Prop. A.2.

## A.1 Profile equations

We substitute (A.3) into (A.1) and write the result as

$$\text{(A.5)}\qquad \sum_{-1}^{M}\epsilon^j\mathcal{F}^j(t,x,Z)|_{Z=\frac{z}{\epsilon}}+\epsilon^M R^{\epsilon,M}(t,x),$$

where we separate $\mathcal{F}^j$ into slow and fast parts

$$\text{(A.6)}\qquad \mathcal{F}^j(t,x,Z)=F^j(t,x)+G^j(t,x_0,Z),$$

and the $G^j$ decrease exponentially to 0 as $Z\to\pm\infty$.

The interior profile equations are obtained by setting the $F^j,G^j$ equal to zero. In the following expressions for $G^j(t,x_0,Z)$, the functions $u^j(t,x)$ and their derivatives are evaluated at $(t,x_0)$. With $W=W(Z,t,x_0,u^0(t,x_0))$ set

$$\text{(A.7)}\qquad \begin{aligned}&\mathbb{L}(t,x_0,Z,\partial_Z)v:=A_{\nu(x_0)}(W)v_Z+(d_uA_\nu(W)\cdot v)W_Z-\\ &\frac{d}{dZ}\left(B_\nu(W)v_Z\right)-\frac{d}{dZ}\left((v\cdot d_uB_\nu(W))W_Z\right),\end{aligned}$$

the operator determined by the linearizing the profile equations about $W$, and

$$\mathcal{L}_0 v := A_0(u^0)v_t + \sum_{j=1}^{d} A_j(u^0)\partial_j v. \tag{A.8}$$

We have

$$\begin{aligned} &F^{-1}(t,x) = 0 \\ &G^{-1}(t,x_0,Z) = A_\nu(W)W_Z - \frac{d}{dZ}(B_\nu(W)W_Z), \end{aligned} \tag{A.9}$$

$$\begin{aligned} &F^0(t,x) = \mathcal{L}_0 u^0 \\ &G^0(t,x_0,Z) = \mathbb{L}(t,x_0,Z,\partial_Z)\mathcal{U}^1 - Q^0(t,x_0,Z), \end{aligned} \tag{A.10}$$

where $Q^0$ decays exponentially as $Z \to +\infty$ and depends only on $(u^0, V^0)$. For $j \geq 1$ we have

$$\begin{aligned} &F^j(t,x) = \mathcal{L}_0 u^j - P^{j-1}(t,x) \\ &G^j(t,x_0,Z) = \mathbb{L}(t,x_0,Z,\partial_Z)\mathcal{U}^{j+1} - Q^j(t,x_0,Z), \end{aligned} \tag{A.11}$$

where $Q^j$ decays exponentially as $Z \to +\infty$ and $P^j$, $Q^j$ depend only on $(u^k, V^k)$ for $k \leq j$. Note that dependence on the cutoff $\chi(x)$ occurs only in the $P^j$.

In writing out the boundary profile equations, we note first that the boundary condition (A.1)(b) is equivalent for $\epsilon > 0$ to

$$\Upsilon(u, \epsilon\partial_T u^2, \epsilon\partial_\nu u^2) = (g_1, g_2, 0). \tag{A.12}$$

With $\mathcal{U}^j(t,x,Z) = (\mathcal{U}^{j,1}, \mathcal{U}^{j,2})$ always evaluated at $(t,x_0,0)$ and

$$\Upsilon'(\mathcal{U}^0)(v,0,v^2_Z) := \begin{pmatrix} \Upsilon'_1(\mathcal{U}^0)v^1 \\ \Upsilon'_2(\mathcal{U}^0)v^2 \\ K_N\partial_Z v^2 \end{pmatrix}, \tag{A.13}$$

the boundary profile equations at order $\epsilon^j$ take the form:

$$\Upsilon(\mathcal{U}^0, 0, \partial_Z\mathcal{U}^{0,2}) = (g_1, g_2, 0) \quad (\text{order } \epsilon^0), \tag{A.14}$$

$$\Upsilon'(\mathcal{U}^0)(\mathcal{U}^1, 0, \partial_Z\mathcal{U}^{1,2}) = (0, 0, c_{1,3}(t,x_0)) \quad (\text{order } \epsilon^1), \tag{A.15}$$

$$\Upsilon'(\mathcal{U}^0)(\mathcal{U}^j, 0, \partial_Z\mathcal{U}^{j,2}) = (c_{j,1}, c_{j,2}, c_{j,3}) \quad (\text{order } \epsilon^j, j \geq 2), \tag{A.16}$$

where the $c_{j,k}(t,x_0)$ depend just on the $\mathcal{U}^p$ and their first derivatives for $p \leq j-1$.

## A.2 Solution of the profile equations

The solution of the profile equations given below assumes transversality of $W(Z, u^0(t, x_0))$ and the uniform Lopatinski condition, as well as the existence of a $K$-family of smooth inviscid symmetrizers. Recall from Lemma 1.23 that the first two conditions both follow from the low frequency uniform Evans condition.

**1.** The interior equations $G^{-1} = 0$ and $F^0 = 0$ and the boundary equation (A.14) are satisfied because of our assumptions about $u^0$ and $W(Z, t, x_0, u^0(t, x_0))$.

**2. Construction of** $(\mathcal{U}^1, u^1)$**.** We construct the functions $\mathcal{U}^1(t, x, Z)$ and $u^1(t, x)$ from the equations $G^0 = 0$, $F^1 = 0$, and the boundary equation (A.15). $\mathcal{U}^1$ will be a sum of three parts

$$\begin{aligned} &\mathcal{U}^1(t, x, Z) = \mathcal{U}^1_a + \mathcal{U}^1_b + \mathcal{U}^1_c, \text{ where} \\ &\mathcal{U}^1_k(t, x, Z) = u^1_k(t, x) + V^1_k(t, x_0, Z),\ k = a, b, c. \end{aligned} \tag{A.17}$$

First use the exponential decay of $Q^0$ to find an exponentially decaying solution $V^1_a(t, x_0, Z)$ to

$$\begin{aligned} &\mathbb{L}(t, x_0, Z, \partial_Z)V^1_a = Q^0(t, x_0, Z) \text{ on } \pm Z \geq 0 \\ &V^1_a \to 0 \text{ as } Z \to +\infty, \end{aligned} \tag{A.18}$$

and define $u^1_a(t, x) \equiv 0$. This problem is easily solved after first conjugating to a constant coefficient ODE using the operators $P$ defined in Lemma 3.1.

Next, for $\mathcal{U}^1_a$ fixed as above, use part (ii) of the definition of transversality (Definition 1.11) to see that we can solve for $\mathcal{U}^1_b(t, x_0, Z) \in \mathcal{S}$ satisfying

$$\begin{aligned} &\mathbb{L}(t, x_0, Z, \partial_Z)\mathcal{U}^1_b = 0 \text{ on } Z \geq 0 \\ &\Upsilon'(\mathcal{U}^0)\left(\mathcal{U}^1_a + \mathcal{U}^1_b, 0, \partial_Z(\mathcal{U}^{1,2}_a + \mathcal{U}^{1,2}_b)\right) = (0, 0, c_{1,3}(t, x_0)). \end{aligned} \tag{A.19}$$

Recalling the definition of $\mathcal{S}$ from Lemma 1.10, we see that $\mathcal{U}^1_b$ has limits as $Z \to \infty$. Define

$$\begin{aligned} &u^1_b(t, x_0) := \lim_{Z\to\infty} \mathcal{U}^1_b(t, x_0, Z), \\ &V^1_b(t, x_0, Z) := \mathcal{U}^1_b(t, x_0, Z) - u^1_b(t, x_0), \end{aligned} \tag{A.20}$$

and let $u^1_b(t, x)$ be any smooth extension of $u^1_b(t, x_0)$ to $[-T_0, T_0] \times \Omega$.

Finally, for an appropriate choice of $u^1_c(t, x_0)$ we need $\mathcal{U}^1_c(t, x_0, Z)$ to satisfy

$$\begin{aligned} &\mathbb{L}(t, x_0, Z, \partial_Z)\mathcal{U}^1_c = 0 \\ &\Upsilon'(\mathcal{U}^0)(\mathcal{U}^1_c, 0, \partial_Z\mathcal{U}^{1,2}_c) = 0 \\ &\lim_{z\to+\infty} \mathcal{U}^1_c(t, x_0, Z) = u^1_c(t, x_0). \end{aligned} \tag{A.21}$$

According to the characterization of $T_q\mathcal{C}(t,x_0)$ given in Remark 1.15, this is possible if and only if $u^1_c(t,x_0)\in T_{u^0(t,x_0)}\mathcal{C}(t,x_0)$. Thus, we first solve for $u^1_c(t,x)$ satisfying the linearized inviscid problem

$$\begin{aligned}&\mathcal{L}_0u^1_c = P^0-\mathcal{L}_0u^1_b\\ &u^1_c(t,x_0)\in T_{u^0(t,x_0)}\mathcal{C}(t,x_0).\end{aligned}\tag{A.22}$$

This problem requires an initial condition in order to be well-posed. The right side in the interior equation of (A.22) is initially defined just for $t\in[-T_0,T_0]$. With a $C^\infty$ cutoff that is identically one in $t\geq -T_0/2$, we can modify the right side to be zero in $t\leq -T_0+\delta$, say. Requiring $u^1_c$ to be identically zero in $t\leq -T_0+\delta$, we thereby obtain a problem for $u^1_c$ that is forward well-posed since $u^0$ satisfies the uniform Lopatinski condition. Thus, there exists a solution to (A.22) on $[-\frac{T_0}{2},T_0]$. This allows us to obtain $\mathcal{U}^1_c(t,x_0,Z))$ satisfying (A.21) and to define

$$V^1_c(t,x_0,Z):=\mathcal{U}^1_c(t,x_0,Z)-u^1_c(t,x_0).\tag{A.23}$$

By construction the functions $(\mathcal{U}^1,u^1)$ satisfy the equations $G^0=0$, $F^1=0$, and the boundary conditions (A.15).

**3. Contruction of $(\mathcal{U}^j,u^j)$, $j\geq 2$.** In the same way, for $j\geq 2$ we use the equations $G^{j-1}=0$, $F^j=0$, and the boundary conditions (A.16) to determine the functions $(\mathcal{U}^j,u^j)$. The corrector $\epsilon^{M+1}u^{M+1}$ is chosen simply to solve away an $O(\epsilon^{M+1})$ error that remains in the boundary conditions after the construction of $\mathcal{U}^M$.

In the next Proposition we formulate a precise statement summarizing the construction of this section. The regularity assertions in the Proposition are justified as in [GMWZ4], Prop. 5.7. Regularity is expressed in terms of the following spaces:

**Definition A.1.** *1. Let $H^s$ (resp. $H^s_b$) be the standard Sobolev space on $[-T_0,T_0]\times\Omega$ (resp. $[-T_0,T_0]\times\partial\Omega$).*

*2. Let $\tilde{H}^s$ be the set of functions $V(t,x_0,Z)$ on $[-T_0,T_0]\times\partial\Omega\times\overline{\mathbb{R}}_+$ such that $V\in C^\infty(\overline{\mathbb{R}}_+,H^s([-T_0,T_0]\times\partial\Omega))$ and satisfies*

$$|\partial^k_ZV(t,x_0,Z)|_{H^s_b}\leq C_{k,s}e^{-\delta|Z|}\text{ for all }k\tag{A.24}$$

*for some $\delta>0$.*

**Proposition A.2** (Approximate solutions). *Assume (H1)-(H6) (with (H4′) replacing (H4) in the symmetric-dissipative case). For given integers $m\geq 0$ and $M\geq 1$ let*

$$s_0>m+\frac{7}{2}+2M+\frac{d+1}{2}.\tag{A.25}$$

*Suppose that the inviscid solution $u^0$ as in* (A.2) *satisfies the uniform Lopatinski condition and that the profiles $W(Z,u^0(t,x_0))$ are transversal. Assume $u^0\in H^{s_0}$ and $u^0|_{\partial\Omega}\in H^{s_0}_b$.*

*Then one can construct $u_a$ as in* (A.3) *satisfying:*

$$\text{(A.26)} \qquad \begin{aligned} &\mathcal{L}_\epsilon u_a = \epsilon^M R^M(t,x) \ \textit{on}\ [-\frac{T_0}{2}, T_0] \times \Omega \\ &\Upsilon(u_a, \partial_T u_a^2, \partial_\nu u_a^2) = (g_1, g_2, 0) \ \textit{on}\ \partial\Omega. \end{aligned}$$

*We have*

$$\text{(A.27)} \qquad u^j(t,x) \in H^{s_0-2j},\ V^j(t,x_0,Z) \in \tilde{H}^{s_0-2j},$$

*and $R^M(t,x)$ satisfies*

$$\text{(A.28)} \qquad \begin{aligned} &(a)\ |(\partial_t, \partial_{x_0}, \epsilon\partial_z)^\alpha R^M|_{L^2} \leq C_\alpha \ \textit{for}\ |\alpha| \leq m + \frac{d+1}{2} \\ &(b)\ |(\partial_t, \partial_{x_0}, \epsilon\partial_z)^\alpha R^M|_{L^\infty} \leq C_\alpha \ \textit{for}\ |\alpha| \leq m. \end{aligned}$$

*In a collar neighborhood of $\partial\Omega$, $\partial_{x_0}$ denotes an arbitrary vector field tangent to $\partial\Omega$ . Away from such a neighborhood, $\partial_{x_0}$ can be a completely arbitrary vector field.*

# B The Tracking Lemma and construction of symmetrizers

In this appendix, we discuss further the high-frequency analysis of Section 3.2, in particular completing the proof of Theorem 3.6 by a treatment of the remaining (much easier) elliptic case. Recall that Theorem 3.6, together with its easy consequences Corollary 3.7 and Proposition 3.8, allows us to reduce to considering only a compact set of frequencies in the stability analysis of many kinds of layers. We applied this result in showing, for example, that Evans stability of small amplitude layers follows from Evans stability of the limiting constant layer.

In the process we prove a useful and previously unremarked relation between the tracking lemma of [ZH, MaZ3, PZ] and the construction of high-frequency symmetrizers as in [MZ1, GMWZ3, GMWZ4, GMWZ6]. In particular, we state the result (used implicitly in [GMWZ6]) that existence of a $k$-family of symmetrizers implies continuity of decaying subspaces in the high-frequency limit (tracking). A similar argument was used in [MZ3] to establish continuity of subspaces in the low-frequency limit.

## B.1 Abstract setting

Consider a generalized resolvent ODE

$$\text{(B.1)} \qquad U' - \check{\mathcal{G}}(z,p,\varepsilon)U = F,$$

where $'$ denotes $\partial_z$ and $(p, \varepsilon)$ comprise frequencies and model parameters, in the limit as $\varepsilon \to 0$. We are interested in the situation, as in the high-frequency regime, that $\check{\mathcal{G}}$ is "slowly varying" in the sense that $\check{\mathcal{G}}'$ is small in an appropriate sense compared to $\check{\mathcal{G}}$.

The following proposition gives one version of this notion in a simple case.

**Proposition B.1.** *(a) Assume the $M \times M$ matrix $\check{\mathcal{G}}(z,p,\varepsilon)$ is a $C^\infty$ function of*

$$(z,p,\varepsilon) \in [0,\infty) \times P \times (0,1] \tag{B.2}$$

*for some parameter set $P$, and that $|\check{\mathcal{G}}| \geq c_1 > 0$ for $c_1$ independent of $(z,p,\varepsilon)$. Suppose*

$$\frac{|\partial_z \check{\mathcal{G}}|}{|\check{\mathcal{G}}|^2} \to 0 \text{ as } \varepsilon \to 0 \text{ uniformly with respect to } (z,p), \tag{B.3}$$

*and suppose*

$$\frac{\mathrm{gap}(\check{\mathcal{G}})}{|\check{\mathcal{G}}|} \geq c_0 > 0 \text{ for } c_0 \text{ independent of } (z,p,\varepsilon). \tag{B.4}$$

*Here,* $\mathrm{gap}(M)$ *denotes the spectral gap of a matrix $M$, defined as the minimum absolute value of the real parts of the eigenvalues of $M$. There exists a conjugating transformation $T(\check{\mathcal{G}})$ taking $\check{\mathcal{G}}$ to block-diagonal form*

$$T\check{\mathcal{G}}T^{-1} = \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix}, \quad \Re M_+ > \eta(z,p,\varepsilon) \quad \text{and} \quad \Re M_- < -\eta(z,p,\varepsilon), \tag{B.5}$$

*where $\eta := \mathrm{gap}(\check{\mathcal{G}}) \geq c_0|\check{\mathcal{G}}|$. For fixed $c_0 > 0$ as in* (B.4) *the matrix $T(\check{\mathcal{G}}(z,p,\varepsilon))$ satisfies*

$$(a)\ |T| \leq C, \quad (b)\ |T^{-1}| \leq C, \quad (c)\ |T_z| \leq C|\check{\mathcal{G}}_z|/|\check{\mathcal{G}}|, \tag{B.6}$$

*uniformly with respect to $(z,p,\varepsilon)$.*

*(b) Setting $U = T^{-1}V$ in* (B.1)*, we obtain*

$$\begin{aligned} V' - \tilde{\mathcal{G}}(z,p,\varepsilon)V &= \tilde{F}, \\ \tilde{\mathcal{G}} &:= \tilde{\mathcal{G}}_{\mathrm{p}} + \Theta(z,p,\varepsilon), \\ \tilde{\mathcal{G}}_{\mathrm{p}} &:= \begin{pmatrix} M_+ & 0 \\ 0 & M_- \end{pmatrix}, \quad |\Theta| \leq \delta, \end{aligned} \tag{B.7}$$

*where $\delta(z,p,\varepsilon) = \delta := C|\partial_z\check{\mathcal{G}}|/|\check{\mathcal{G}}|$, and so*

$$\delta/\eta \to 0 \text{ as } \varepsilon \to 0 \text{ uniformly with respect to } (z,p). \tag{B.8}$$

*Proof.* **(a).** Consider the matrix $\mathcal{K} := \check{\mathcal{G}}/|\check{\mathcal{G}}|$ which has a spectral gap uniformly bounded away from 0 by assumption. It follows by standard matrix perturbation theory [Kat] that there exists a smooth transformation $T(\mathcal{K})$ taking $\mathcal{K}$ to block-diagonal form

$$T\mathcal{K}T^{-1} = \begin{pmatrix} N_+ & 0 \\ 0 & N_- \end{pmatrix}, \quad \Re N_+ > c_0 \quad \text{and} \quad \Re N_- < -c_0 \tag{B.9}$$

and satisfying (B.6)(a), (b). Clearly the same $T$ satisfies (B.5). Moreover, with obvious notation we have

$$|T_z| \leq |dT/d\mathcal{K}|\ |d\mathcal{K}/d\check{\mathcal{G}}|\ |\check{\mathcal{G}}_z| \leq C|\check{\mathcal{G}}_z|/|\check{\mathcal{G}}|. \tag{B.10}$$

**(b).** Defining $V$ by $U = T^{-1}V$, substituting in (B.1), and using (B.9) yields (B.7) with

$$\tilde{F} = TF, \quad \Theta = -T(T^{-1})_z, \tag{B.11}$$

so by (B.6) $|\Theta| \leq C|\check{\mathcal{G}}_z|/|\check{\mathcal{G}}|$. Thus, (B.8) follows from the assumption (B.3).

□

For example, in the application of Proposition B.1 to the "elliptic zone" in section B.4.1, we will have

$$\varepsilon = |\zeta|^{-1}, \ |\check{\mathcal{G}}| \sim |\zeta|, \ |\check{\mathcal{G}}_z| \sim |\zeta|, \text{ and so } \delta \sim 1, \ \ \eta \sim |\zeta|. \tag{B.12}$$

Depending on the application, $\eta$ may be bounded, go to zero, or go to infinity as $\varepsilon \to 0$, and likewise $|\check{\mathcal{G}}|$. In some sense this is artificial, since the scale of $\check{\mathcal{G}}$ (and thus of $\eta$) may be changed arbitrarily by rescaling the independent variable $z$; however, it is convenient not to have to rescale.

More generally, following [MaZ3], we take the existence of a transformation to form (B.7)–(B.8) as *defining* the notion of a slowly-varying coefficient $\mathcal{G}$. The construction of such transformations must in general be done quite carefully by hand, and does not follow by a simple argument like that of Proposition B.1; see for example the treatment in Section 7, [GMWZ6], and here in the proof of Theorem 3.6 of the general case away from the elliptic zone. Proposition B.1 suffices to treat strictly parabolic systems, as pointed out in [GZ, ZH, Z3].

## B.2 Tracking

For systems (B.1) as above, we have the following version of the "tracking lemma" of [MaZ3]. The lemma implies that in the modified coordinates of (B.7), under assumption (B.8), the subspace of initial data at $z_0 \in \mathbb{R}$ of decaying (resp. growing) solutions of the homogeneous equation approximately "tracks" the stable (resp. unstable) subspace of the principal part $\tilde{\mathcal{G}}_{\mathrm{p}} = \mathrm{blockdiag}\{M_+, M_-\}$ evaluated at $z_0$ in the sense that one subspace approaches the other uniformly as $\varepsilon \to 0$.

In view of (B.8) we will often suppress the dependence of $\delta$ and $\eta$ on $(z, p)$ in what follows and simply write $\delta(\varepsilon)$, $\eta(\varepsilon)$.

**Proposition B.2** ([MaZ3]). *Consider an approximately diagonalized system* (B.7) *with* $\tilde{F} \equiv 0$ *satisfying bound* (B.8).

*(i) For all* $0 < \epsilon \leq \epsilon_0$, *there exist (unique) linear transformations* $\Phi_1^\epsilon(z, p)$ *and* $\Phi_2^\epsilon(z, p)$, *with* $C^\infty$ *dependence on* $(z, p, \varepsilon)$ *for which the graphs*

$$\{(Z_1, \Phi_2^\epsilon(z, p)Z_1)\} \text{ and } \{(\Phi_1^\epsilon(z, p)Z_2, Z_2)\}$$

*are invariant under the flow of* (B.7)*. The graphs consist precisely of the initial data at* $z = z_0$ *of solutions of* (B.7) *that are respectively exponentially growing and decaying.*

*Moreover, the functions* $\Phi_j^\epsilon$ *satisfy*

$$|\Phi_1^\epsilon|,\, |\Phi_2^\epsilon| \le C\delta(\epsilon)/\eta(\epsilon) \text{ for all } z.$$

*(ii)In particular, the subspace* $E_-(p,\varepsilon)$ *of data at* $z = 0$ *for which the solution of* (B.7) *decays as* $z \to +\infty$ *is given by the graph* $\{(\Phi_1^\varepsilon(0,p)v_-, v_-) : v_- \in \mathbb{C}^{dimM_-}\}$ *and converges as* $\varepsilon \to 0$ *to* $\tilde{E}_- := \{(0, v_-) : v_- \in \mathbb{C}^{dimM_-}\}$.

**Proof.** This can be proved by a contraction mapping argument carried out on the "lifted" equations governing the flow of the conjugating matrices $\Phi_j^\varepsilon$; see Appendix C, [MaZ3] for details. Proposition B.4 provides an alternative proof of part(ii), which is the only part we use in this paper, in the case that $\tilde{\mathcal{G}}$ exponentially approaches a limit as $z \to +\infty$ for each fixed $p$, $\varepsilon$ (not necessarily uniformly), which holds always in our applications. A related proof based on energy estimates/invariant cones appears in [ZH, Z1].

### B.3 Symmetrizers

With the same initial preparations (i.e., reduction to form (B.7)), one may also obtain directly bounds on the inhomogeneous resolvent equation by the method of Kreiss symmetrizers [K, MZ1]. Consider (B.1) on $[0, +\infty)$, augmented with some specified boundary condition

$$\Gamma(p,\varepsilon)U = G, \text{ where } |\Gamma(p,\varepsilon)| \le C, \tag{B.13}$$

uniformly for $p \in P$, $\varepsilon \in (0,1]$. The corresponding boundary condition for the conjugated problem (B.7) is then

$$\tilde{\Gamma}(p,\varepsilon)V := \Gamma(p,\varepsilon)T^{-1}(0,p,\varepsilon)V = G. \tag{B.14}$$

Defining $\tilde{E}_-(p,\varepsilon)$ as the stable subspace of the principal part blockdiag$\{M_+, M_-\}$ evaluated at $z_0 = 0$ (i.e., $\tilde{E}_- := \{(0,v) : v \in \mathbb{C}^{dimM_-}\}$) and defining the "frozen-coefficients Evans function"

$$\tilde{D}(p,\varepsilon) := \det(\tilde{E}_-, \ker \tilde{\Gamma}), \tag{B.15}$$

we have the following result.

**Proposition B.3.** *Consider the problem* (B.1), (B.13) *under the assumptions of Proposition B.2. Assume this problem is "frozen-coefficients stable" in the sense that* $|\tilde{D}| \ge c_0 > 0$ *for* $\varepsilon > 0$ *sufficiently small, uniformly with respect to* $p \in P$. *Then for some* $C > 0$ *that can be taken independent of* $p \in P$ *and* $\varepsilon$ *sufficiently small,*

$$\sqrt{\eta}\|U\|_{L^2(\mathbb{R}^+)} + |U(0)| \le C(\|\tilde{F}\|_{L^2(\mathbb{R}^+)}/\sqrt{\eta} + |G|). \tag{B.16}$$

*Proof.* **1.** In view of the properties (B.5), (B.6) of the conjugating transformation $T$, it suffices to prove the estimate (B.16) for $V$ satisfying the conjugated problem (B.7) and the boundary condition (B.14).

**2.** Defining $S_k := \text{blockdiag}\{k\text{Id}, -\text{Id}\}$, $k \geq 1$, we have for $\varepsilon > 0$ sufficiently small,

$$\begin{aligned}
&(i) S_k = S_k^*, \quad |S_k| \leq k;\\
&(ii) \Re S_k \tilde{\mathcal{G}}_p \geq \eta(\varepsilon);\\
&(iii) S_k v \cdot v \geq k|\Pi_+ v|^2 - |\Pi_- v|^2,
\end{aligned}$$

Here $\Pi_\pm$ are the projections onto the unstable and stable subspaces of $\tilde{\mathcal{G}}_p(0,p,\varepsilon)$. Writing $v = (v_+, v_-)$ we have

$$\Pi_+ v = (v_+, 0), \quad \Pi_- = (0, v_-). \tag{B.17}$$

Thus $S_k$ is a $k$-family of symmetrizers in the sense of [GMWZ6].

Taking the real part of the $L^2$ inner product of $-S_k V$ with (B.7) and integrating by parts, we obtain

$$\begin{aligned}
(1/2) S_k V(0) \cdot V(0) + \langle V, \Re(S_k \tilde{\mathcal{G}}) V \rangle &= \Re\langle -S_k V, \Theta V\rangle + \Re\langle -S_k V, \tilde{F}\rangle\\
&\leq |S_k| (\delta \|V\|^2 + \|V\| \|\tilde{F}\|).
\end{aligned}$$

From this we may deduce using (i)–(iii) and $\delta/\eta \to 0$ that, for $\varepsilon > 0$ sufficiently small,

$$k|\Pi_+ V(0)|^2 + (\eta/2)\|V\|^2 \leq k\|V\|\|\tilde{F}\| + |\Pi_- V(0)|^2. \tag{B.18}$$

But $|\det(\tilde{E}_-, \ker \tilde{\Gamma})| \geq c_0$ implies that $|\tilde{\Gamma} v| \geq c|v|$ for $v \in \tilde{E}_-$, where $c = c(c_0, |\tilde{\Gamma}|) > 0$. Thus,

$$|\Pi_- V(0)| \leq |\tilde{\Gamma}\Pi_- V(0)| \leq |\tilde{\Gamma} V(0)| + |\tilde{\Gamma}\Pi_+ V(0)| \leq |G| + |\tilde{\Gamma}||\Pi_+ V(0)|.$$

Combining with (B.18), taking $k$ sufficiently large relative to $|\tilde{\Gamma}|$, and using Young's inequality to bound $k\|\tilde{F}\|\|V\| \leq (\eta/4)\|V\|^2 + (|k|^2/\eta)\|\tilde{F}\|^2$, we obtain the result. ☐

The next Proposition (more precisely, its proof) formulates the new observation that the conclusion of part (ii) of Proposition B.2 is a consequence of the estimate (B.16); that is, information provided by the tracking lemma may be deduced as a consequence of the basic symmetrizer construction.

**Proposition B.4.** *Consider an $M \times M$ system* (B.1) *satisfying the assumptions of Proposition B.2. Let $\mathbb{E}_-(p,\varepsilon)$ denote the subspace of initial data at $z = 0$ for which the solution of*

$$U' - \check{\mathcal{G}}(z,p,\varepsilon)U = 0 \tag{B.19}$$

*decays to* $0$ *as $z \to \infty$. Let $\tilde{\mathbb{E}}_-(p,\varepsilon)$ denote the stable subspace of $\check{\mathcal{G}}(0,p,\varepsilon)$. Assuming*

$$\dim \mathbb{E}_-(p,\varepsilon) = \dim \tilde{\mathbb{E}}_-(p,\varepsilon) \text{ for } p \in P, \ \varepsilon \text{ small.} \tag{B.20}$$

*we have*

$$\mathbb{E}_-(p,\varepsilon) \to \tilde{\mathbb{E}}_-(p,\varepsilon) \text{ as } \varepsilon \to 0. \tag{B.21}$$

**Remark B.5.** *Condition* (B.20) *holds, for example, if $\tilde{\mathcal{G}}$ exponentially approaches a limit as $z \to +\infty$ for each fixed p, $\varepsilon$ (not necessarily as $z \to +\infty$, by the conjugation lemma, Lemma 3.1. This is always the case for our applications here.*

*Proof.* **1.** Let $E_-(p,\varepsilon)$ and $\tilde{E}_-(p,\varepsilon)$ be the spaces appearing in Proposition B.2. They are the exact analogues of $\mathbb{E}_-$ and $\tilde{\mathbb{E}}_-$ for the conjugated system $V' - \tilde{\mathcal{G}}V = 0$. From (B.5) and the properties of the conjugator $T$ we have

$$\begin{aligned} \mathbb{E}_-(p,\varepsilon) &= T^{-1}(0,p,\varepsilon)E_-(p,\varepsilon) \\ \tilde{\mathbb{E}}_-(p,\varepsilon) &= T^{-1}(0,p,\varepsilon)\tilde{E}_-(p,\varepsilon), \end{aligned} \tag{B.22}$$

so it is equivalent to show

$$E_-(p,\varepsilon) \to \tilde{E}_-(p,\varepsilon) = \{(0,v_-) : v_- \in \mathbb{C}^{dimM_-}\} \text{ as } \varepsilon \to 0. \tag{B.23}$$

**2.** The proof depends on the fact that the estimate (B.16) holds for *any* boundary condition satisfying the hypotheses of Proposition B.3. Let $W \subset \mathbb{C}^M$ be any subspace *transverse* to the fixed space $\tilde{E}_-(p,\varepsilon)$ and such that $\dim W^\perp = \dim \tilde{E}_-(p,\varepsilon)$. Defining $\Gamma_W$ to be orthogonal projection onto $W^\perp$, we have

$$\ker \Gamma_W = W, \quad \text{rank}\, \Gamma_W = \dim \tilde{E}_-(p,\varepsilon). \tag{B.24}$$

Since $|\det(\tilde{E}_-, \ker \Gamma_W)| \geq c_0 > 0$, the hypotheses of Proposition B.3 are satisfied by the conjugated system (B.7) with the boundary condition $\Gamma_W$. Assuming that $V$ is any decaying solution of $V' - \tilde{\mathcal{G}}V = 0$, we have $V(0) \in E_-(p,\varepsilon)$. From the estimate (B.16) for the conjugated problem with $\tilde{F} = 0$ we deduce

$$|V(0)| \leq C|\Gamma_W V(0)| \text{ for } \varepsilon \text{ sufficiently small.} \tag{B.25}$$

By (B.20) and Lemma 1.21 this implies

$$|\det(E_-(p,\varepsilon), \ker \Gamma_W| \geq c > 0, \tag{B.26}$$

where $c$ depends on $C$ and $|\Gamma_W| = 1$. Therefore, $W$ is also transverse to $E_-(p,\varepsilon)$ for $\varepsilon$ sufficiently small. Since we are free to make different choices of $W$ tranverse to $\tilde{E}(p,\varepsilon)$, this can only be true if (B.21) holds.

□

As an immediate consequence of Proposition B.3 and Proposition B.2, part (ii), we obtain

**Corollary B.6.** *Suppose* (B.1) *satisfies the assumptions of Proposition B.2, and let*

$$\mathbb{D}(p,\varepsilon) := \det(\mathbb{E}_-, \ker \Gamma). \tag{B.27}$$

*Then* (B.1), (B.13) *is "uniformly Evans stable" in the sense that* $|\mathbb{D}| \geq c_0 > 0$ *for* $\varepsilon > 0$ *sufficiently small if and only if* (B.1), (B.13) *is "frozen coefficients" stable in the sense that*

$$\tilde{\mathbb{D}}(p,\varepsilon) := \det(\tilde{\mathbb{E}}_-, \ker \Gamma) \geq c_1 > 0 \tag{B.28}$$

*for $\varepsilon$ sufficiently small. In either case the estimate* (B.16) *holds.*

**Remark B.7.** *In this presentation, we have subsumed large parts of the usual symmetrizer construction into the preparatory transformations to the approximately block-diagonal form* (B.7)*. One of the main points of this development is to demonstrate that tracking and symmetrizer estimates are essentially automatic once we can reduce a system to form* (B.7)*–*(B.8)*. Moreover, all high-frequency estimates derived in [ZH, Z3, GMWZ4, GMWZ6] were either obtained originally in this way, or may be rephrased in this form.*

**Remark B.8.** *Another consequence of Corollary B.6 is that uniform high-frequency Evans stability implies the maximal estimate* (3.8) *for $\zeta \in \mathcal{E}$. Indeed, the estimate* (B.16) *is equivalent to* (3.8) *in this application.*

## B.4 Application to High Frequencies: Proof of Theorem 3.6

We conclude by illustrating through explicit computations the application of these methods to the high-frequency analysis of Section 3.2.

Recall from (1.43) the linearized eigenvalue equation $Lu = f$, where

$$L = -\mathcal{B}(z)\partial_z^2 + \mathcal{A}(z,\zeta)\partial_z + \mathcal{M}(z,\zeta) \tag{B.29}$$

with coefficients given by

$$\begin{aligned}
\mathcal{B}(z) &= B_{dd}(w(z))\\
\mathcal{A}(z,\zeta) &= A_d(w(z)) - \sum_{j=1}^{d-1} i\eta_j\big(B_{jd} + B_{d,j}\big)(w(z)) + E_d(z)\\
\mathcal{M}(z,\zeta) &= (i\tau+\gamma)A_0(w(z)) + \sum_{j=1}^{d-1} i\eta_j\big(A_j(w(z)) + E_j(z)\big)\\
&\qquad + \sum_{j,k=1}^{d-1} \eta_j\eta_k B_{j,k}(w(z)) + E_0(z).
\end{aligned} \tag{B.30}$$

The $E_k$ are functions independent of $\zeta$ which involve derivatives of $w$ and thus converge to 0 at an exponential rate when $z$ tends to infinity. Moreover, we note that

$$E_k^{11} = 0, \quad E_k^{12} = 0 \quad \text{for } k > 0. \tag{B.31}$$

With (1.2), we also remark that $\mathcal{M}^{12}$ does not depend on $\tau$ and $\gamma$.

As in (1.45) we rewrite the linearized problem as a first-order system

$$\partial_z U - \mathcal{G}(z,\zeta)U = F, \quad \Gamma(\zeta)U_{|z=0} = G, \tag{B.32}$$

$$\mathcal{G} = \begin{pmatrix} \mathcal{G}^{11} & \mathcal{G}^{12} & \mathcal{G}^{13}\\ 0 & 0 & \mathrm{Id}\\ \mathcal{G}^{31} & \mathcal{G}^{32} & \mathcal{G}^{33} \end{pmatrix}, \tag{B.33}$$

where $U = (u, \partial_z u^2) = (u^1, u^2, \partial_z u^2) \in \mathbb{C}^{N+N'}$ and $\zeta = (\gamma, \tau, \eta)$.

Here,

$$\begin{aligned} \mathcal{G}^{11} &= -(\mathcal{A}^{11})^{-1}\mathcal{M}^{11}, & \mathcal{G}^{31} &= (\mathcal{B}^{22})^{-1}(\mathcal{A}^{21}\mathcal{G}^{11} + \mathcal{M}^{21}), \\ \mathcal{G}^{12} &= -(\mathcal{A}^{11})^{-1}\mathcal{M}^{12}, & \mathcal{G}^{32} &= (\mathcal{B}^{22})^{-1}(\mathcal{A}^{21}\mathcal{G}^{12} + \mathcal{M}^{22}), \\ \mathcal{G}^{13} &= -(\mathcal{A}^{11})^{-1}\mathcal{A}^{12}, & \mathcal{G}^{33} &= (\mathcal{B}^{22})^{-1}(\mathcal{A}^{21}\mathcal{G}^{13} + \mathcal{A}^{22}), \end{aligned}$$

Note that $\mathcal{G}^{11}$, $\mathcal{G}^{12}$, $\mathcal{G}^{31}$ and $\mathcal{G}^{33}$ are first order (linear or affine in $\zeta$), that $\mathcal{G}^{32}$ is second order (at most quadratic in $\zeta$) and that $\mathcal{G}^{13}$ is of order zero (independent of $\zeta$). We denote by $\mathcal{G}^{ab}_{\mathrm{p}}$ their principal part (leading order part as polynomials). We note that

$$\mathcal{G}^{ab}_{\mathrm{p}}(z, \zeta) = G^{ab}_{\mathrm{p}}(w(z), \zeta) \quad \text{when } (a, b) \neq (3, 1), \tag{B.34}$$

with

$$\begin{array}{ll} G^{11}_{\mathrm{p}}(u, \zeta) & = -(A^{11}_d(u))^{-1}\big((\gamma + i\tau)A^{11}_0(u) + \sum_{j=1}^{d-1} i\eta_j A^{11}_j(u)\big), \\ G^{12}_{\mathrm{p}}(u, \zeta) & = -(A^{11}_d(u))^{-1}\sum_{j=1}^{d-1} i\eta_j A^{12}_j(u) \\ G^{13}_{\mathrm{p}}(u) & = -(A^{11}_d(u))^{-1}A^{12}_d(u) \\ G^{32}_{\mathrm{p}}(u, \zeta) & = (B^{22}(u))^{-1}\sum_{j,k=1}^{d-1} \eta_j\eta_k B^{22}_{j,k}(u)\big), \\ G^{33}_{\mathrm{p}}(u, \zeta) & = -(B^{22}(u))^{-1}\sum_{j=1}^{d-1} i\eta_j\big(B^{22}_{j,d}(u) + B^{22}_{d,j}(u)\big). \end{array}$$

The principal term of $\mathcal{G}^{3,1}$ involves derivatives of the profile $w$. Denoting by $p = \lim_{z\to+\infty} w(z) = w(\infty)$ the end state of the profile $w$, we note that the end state of $\mathcal{G}^{31}_{\mathrm{p}}$ is

$$\mathcal{G}^{31}_{\mathrm{p}}(\infty, \zeta) = (B^{22}(p))^{-1}\Big((\gamma + i\tau)A^{21}_0(p) + \sum_{j=1}^{d-1} i\eta_j A^{21}_j(p) + A^{21}_d(p)G^{11}_{\mathrm{p}}(p, \zeta)\Big).$$

There are similar formulas using the matrices $\overline{A}_j$ and $\overline{B}_{j,k}$ of (1.5).

### B.4.1 The elliptic zone

We now prove Theorem 3.6 in the easiest case, which is for frequencies $\zeta$ lying in the *elliptic zone*

$$\mathcal{E} := \{(\tau, \gamma, \eta) : \gamma \geq \delta|\zeta| \text{ and } |\eta| \geq \delta|\zeta|\} \text{ with } \delta > 0, \tag{B.35}$$

with $|\zeta|$ sufficiently large. (Recall that this is the final remaining case not treated in Section 3.2.)

In this case, the factors $(1+\gamma)$ and $\Lambda(\zeta) = \big(\tau^2 + \gamma^2 + |\eta|^4\big)^{1/4}$ appearing in scaling (3.11) are both of order $|\zeta|$, and so we may replace (3.11) by the simpler rescaling

$$\begin{aligned} J_\zeta(u^1, u^2, u^3) &:= \big(|\zeta|^{\frac{1}{2}}u^1, |\zeta|^{\frac{1}{2}}u^2, |\zeta|^{-\frac{1}{2}}u^3\big) := \check{U} \\ J_\zeta(g^1, g^2, g^3) &:= \big(|\zeta|^{\frac{1}{2}}g^1, |\zeta|^{\frac{1}{2}}g^2, |\zeta|^{-\frac{1}{2}}g^3\big) := \check{G}. \end{aligned} \tag{B.36}$$

If we define

$$\Gamma_e^{sc}(u^1,u^2,u^3) := (\Gamma_1 u^1, \Gamma_2 u^2, K_d u^3 + |\zeta|^{-1} K_T(\eta) u^2), \tag{B.37}$$

then (B.32) may be written equivalently as

$$\partial_z \check{U} - \check{\mathcal{G}}(z,\zeta)\check{U} = \check{F}, \quad \Gamma_e^{sc}(\zeta)\check{U} = \check{G} \tag{B.38}$$

where

$$\check{\mathcal{G}} = \begin{pmatrix} \mathcal{G}^{11} & \mathcal{G}^{12} & |\zeta|\mathcal{G}^{13} \\ 0 & 0 & |\zeta|\mathrm{Id} \\ |\zeta|^{-1}\mathcal{G}^{31} & |\zeta|^{-1}\mathcal{G}^{32} & \mathcal{G}^{33} \end{pmatrix} := \begin{pmatrix} \mathcal{G}^{11} & \mathcal{P}^{12} \\ \mathcal{P}^{21} & \mathcal{P}^{22} \end{pmatrix} \tag{B.39}$$

with obvious definitions of $\mathcal{P}^{ab}$. Note that $\check{\mathcal{G}}$ is of order one, while $\mathcal{P}^{21}$ is of order zero. Thus

$$\check{\mathcal{G}}(z,\zeta) = \check{\mathcal{G}}_{\mathrm{p}}(z,\zeta) + O(1), \quad \check{\mathcal{G}}_{\mathrm{p}} = \begin{pmatrix} \mathcal{G}^{11}_{\mathrm{p}} & \mathcal{P}^{12}_{\mathrm{p}} \\ 0 & \mathcal{P}^{22}_{\mathrm{p}} \end{pmatrix} = O(|\zeta|), \tag{B.40}$$

where the principal part $\check{\mathcal{G}}_{\mathrm{p}}$ is in fact homogeneous degree one in $|\zeta|$, and depends on $z$ only through the profile $w(z)$ and not its derivatives.

Consider now the Fourier-Laplace transform of the upper triangular system

$$\begin{aligned} &u^1_t + \sum_j \overline{A}^{11}_j(w(0))\partial_j u^1 + \sum_j \overline{A}^{12}_j(w(0))\partial_j u^2 = 0, \\ &u^2_t - \sum_{j,k} \overline{B}^{22}_{jk}(w(0))\partial_j\partial_k u^2 = 0 \end{aligned} \tag{B.41}$$

written as a first-order system:

$$\partial_z U - \mathcal{G}_{ut}(0,\zeta)U = 0. \tag{B.42}$$

With $\check{U}$ as above observe that we can write (B.42) equivalently as

$$\partial_z \check{U} - \check{\mathcal{G}}_{\mathrm{p}}(0,\zeta)\check{U} = 0. \tag{B.43}$$

**Proposition B.9.** *Suppose $\zeta$ lies in the elliptic zone. Let $\check{\mathbb{E}}_-(\zeta)$ denote the space of initial data at $z=0$ of decaying solutions of $\partial_z \check{U} - \check{\mathcal{G}}(z,\zeta)\check{U} = 0$, and let $\tilde{\mathbb{E}}_-(\zeta)$ be the stable subspace of $\check{\mathcal{G}}_p(0,\zeta)$. Then*

$$\check{\mathbb{E}}_-(\zeta) \to \tilde{\mathbb{E}}_-(\zeta) \text{ as } |\zeta| \to \infty. \tag{B.44}$$

*Proof.* With $p := |\zeta|^{-1}\zeta$ and $\varepsilon := |\zeta|^{-1}$ we can write (with slight abuse)

$$\check{\mathcal{G}}(z,\zeta) = \check{\mathcal{G}}(z,p,\varepsilon) \text{ and } \check{\mathcal{G}}_{\mathrm{p}}(z,\zeta) = \check{\mathcal{G}}_{\mathrm{p}}(z,p,\varepsilon). \tag{B.45}$$

The system (B.38) thus has the form (B.1). By Lemma 7.3(i) of [GMWZ6]) the matrix $\check{\mathcal{G}}(z,p,\varepsilon)$ has a spectral gap $\eta(\varepsilon) \sim |\zeta| = \frac{1}{\varepsilon}$. Since

$$|\check{\mathcal{G}}(z,\zeta)| \sim |\zeta| \text{ and } |\check{\mathcal{G}}_z(z,\zeta)| \sim |\zeta|, \tag{B.46}$$

Proposition B.1 implies there is a conjugator $T$ satisfying (B.5),(B.6). Hence $\delta(\varepsilon) \sim 1$ by (B.10),(B.11), and thus $\delta/\eta \to 0$ as $\varepsilon \to 0$.

By Remark B.5 we can now apply Proposition B.4 to conclude that $\check{\mathbb{E}}_-(\zeta)$ approaches the stable subspace of $\check{\mathcal{G}}(0,p,\varepsilon)$ as $\varepsilon \to 0$. The $O(1)$ term in (B.40) introduces an $O(\varepsilon)$ difference between the stable subspace of $\check{\mathcal{G}}(0,p,\varepsilon)$ and that of $\check{\mathcal{G}}_{\mathrm{p}}(0,p,\varepsilon)$ (there is an $O(\varepsilon)$ difference between the corresponding conjugators $T$), so we conclude (B.44). $\square$

*Proof of Theorem 3.6 for the elliptic zone.* Here we use the notation of the previous Proposition.

**1.** For $\zeta \in \mathcal{E}$ and $\Gamma_e^{sc}$ as in (B.38) define a slightly modified rescaled Evans function

$$D_e^{sc}(\zeta) = \det(\check{\mathbb{E}}_-(\zeta), \ker \Gamma_e^{sc}). \tag{B.47}$$

Using Lemma 1.21 in the same argument that showed the equivalence of estimate (3.10) and the uniform Evans stability condition Definition 3.4, we see that for $\zeta \in \mathcal{E}$,

$$\begin{aligned} &\text{there exist } c_0, R \text{ such that } |D^{sc}(\zeta)| \geq c_0 \text{ for } |\zeta| \geq R \Leftrightarrow \\ &\text{there exist } c_1, R' \text{ such that } |D_e^{sc}(\zeta)| \geq c_1 \text{ for } |\zeta| \geq R'. \end{aligned} \tag{B.48}$$

**2.** Defining a frozen-coefficient Evans function

$$\tilde{D}^{sc}(\zeta) = \det(\tilde{\mathbb{E}}_-(\zeta), \ker \Gamma_e^{sc}), \tag{B.49}$$

we conclude from Corollary B.6 and Proposition B.9 that the conditions (B.48) are equivalent to

$$\text{there exist } c_2, R'' \text{ such that } |\tilde{D}^{sc}(\zeta)| \geq c_2 \text{ for } |\zeta| \geq R''. \tag{B.50}$$

Recall that $\tilde{\mathbb{E}}_-(\zeta)$ is the stable subspace of $\check{\mathcal{G}}_{\mathrm{p}}(0,\zeta)$, which defines the problem (B.43) equivalent to the upper triangular system (B.41).

**3.** To complete the proof we write

$$\Gamma_e^{sc} = (\Gamma_1, \Gamma_*^{sc}) \tag{B.51}$$

and observe that

$$\ker \Gamma_e^{sc} = \begin{pmatrix} \ker \Gamma_1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ \ker \Gamma_*^{sc} \end{pmatrix}$$

From the form of $\check{\mathcal{G}}_{\mathrm{p}}$ we see that $\tilde{\mathbb{E}}_-(\zeta)$ consists of vectors of the form

$$\begin{pmatrix} v_a \\ 0 \end{pmatrix}, \; v_a \in S(\mathcal{G}_{\mathrm{p}}^{11}) \text{ and } \begin{pmatrix} * \\ v_b \end{pmatrix}, \; v_b \in S(\mathcal{P}_{\mathrm{p}}^{22}), \tag{B.52}$$

where $S(M)$ denotes the stable subspace of $M$. After performing obvious column operations on $\tilde{D}^{sc}(\zeta)$ to express the determinant in upper block triangular form, we thus obtain (up to a sign)

$$\tilde{D}^{sc}(\zeta) = \det\Big(S(\mathcal{G}_{\mathrm{p}}^{11}), \ker\Gamma_1\Big) \times \det\Big(S(\mathcal{P}_{\mathrm{p}}^{22}), \ker\Gamma_*^{sc}\Big). \tag{B.53}$$

Although $\Gamma_*^{sc}(\zeta)$ here differs slightly from $\Gamma_*^{sc}(\zeta)$ in (3.20), the argument that established the equivalence (B.48) allows us to conclude that $D^2(\zeta)$ (3.20) is bounded away from 0 for $|\zeta|$ large if and only if the second factor on the right in (B.53) is. The first factor equals $D^1(\zeta)$ (3.20), so this completes the proof.

□

**Remark B.10.** *For coupled boundary conditions, the above analysis shows that the high-frequency Evans condition is more complicated, involving at least the upper triangular system* (B.41) *rather than the decoupled* (1.8).

### B.4.2 The general case

In the remaining frequency regimes

$$C_\delta := \{\zeta : 0 \le \gamma \le \delta|\zeta|\} \cup \{\zeta : |\eta| \le \delta|\zeta|\},$$

$\delta > 0$ sufficiently small, the reduction to form (B.7) is considerably more complicated, in particular involving multiplication of hyperbolic modes $u^1$ by an exponential weighting function to obtain definiteness of $M_\pm$. Moreover, the resulting terms $M_\pm$ depend in hyperbolic modes both on derivatives of the profile and on the chosen exponential weight. Thus, it is no longer true that the effective principal part $\tilde{G}$ after rescaling/appropriate coordinate transformations involves only frozen coefficients of the triangular system (B.41) as in the previous case. *However, it is still true that the associated stable (resp. unstable) subspaces of $\tilde{G}$ depend only on those coefficients*: indeed, only on the coefficients of the fully decoupled system (3.17). Thus, the above conclusions about the rescaled Evans function and uniform high-frequency stability (based on the form of the stable subspaces and not on $M_\pm$) hold true in this case as well. The maximal estimates follow likewise from Proposition B.3 as before, once form (B.7) has been achieved. For detailed treatments, including in particular an implicit reduction to form (B.7), see Section 3.2 of this paper and Sections 7.2–7.4 of [GMWZ6].

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